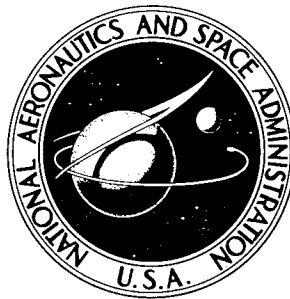


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# A GENERAL MATHEMATICAL MODEL FOR BEAM AND PLATE VIBRATION IN BENDING MODES USING LUMPED PARAMETERS

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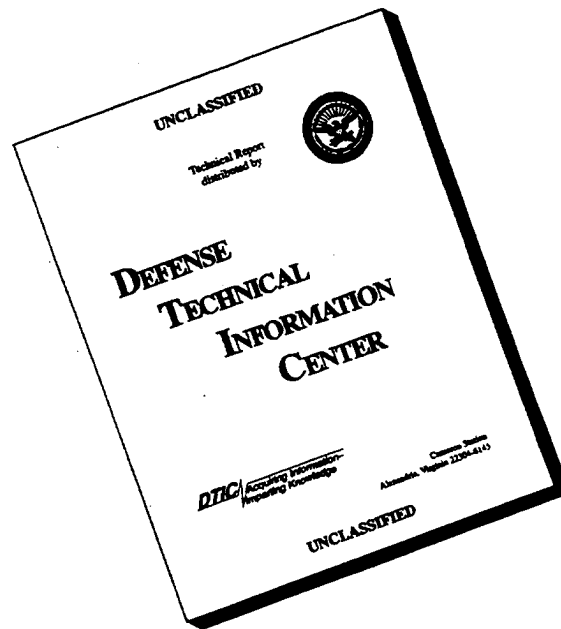
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By Irvin P. Vatz \*

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## A GENERAL MATHEMATICAL MODEL FOR BEAM AND PLATE VIBRATION IN BENDING MODES USING LUMPED PARAMETERS

### SUMMARY

The analytic method proposed in this paper solves for natural frequencies and mode shapes of distributed mass and stiffness systems such as beams and plates. The structure is divided into an equivalent system of discrete masses and springs. A schematic diagram can be built up to show the nature of each item affecting the system. Increased off-hand evaluation of design changes results. A group of simultaneous equations is made up from the discrete schematic system. These equations comprise the mathematical model of the system.

The mathematical model of the beam is completely general. The model is a matrix equation which will accept any input, any restraint, or any combination of end conditions. One model will suffice for all beam problems. Damping can be considered, but is not included in the main thesis of the paper. The model is in sufficient detail to accept the particulars of a nonuniform beam where mass and stiffness can be broken into lumped discrete but unequal values. The number of masses limits the eigenvalue solutions. Reasonable accuracy is dependent on a minimum of three masses between nodes. This is equivalent to four masses per modal number, plus one for the end node. Thus a system with seventeen masses would be reasonable for the calculation of the first four modes of a beam.

Flat plates are simulated as two sets of cross-coupled beams. A nonuniform plate model is developed. End conditions of the individual beams may be varied, masses may be equated to zero, etc, to set up a rather complicated model. As in the beam model, the plate model is very general and only one model is necessary where the size is large enough to accept the required detail. Matrix size limits the number of eigenvalue solutions available. The model of a cross-coupled plate of sixteen masses is derived in the paper. Actually, practical solutions would require a much larger equation.

The model does not account for any extensional deformation. Only bending is considered. The application of the method to shells and other curved plates is suggested when only bending modes need to be considered.

## SECTION I. INTRODUCTION

Mechanical systems comprised of distributed mass and distributed stiffness have an infinite number of normal modes of vibration. Beams are characteristic of the distributed systems and have only one set of normal modes per single direction of deflection. They therefore are representative of the simplest approach to distributed system dynamic analysis. Other systems such as plates and shells can be represented by a system of cross-coupled beams once a suitable system of impedance and coordinate designation is selected.

Systems comprised of uniformly distributed mass and stiffness lend themselves to the usual methods of solution, by utilizing some form of the Rayleigh method by equating maximum strain to maximum kinetic energy for each mode. These methods arrive at mode shapes and natural frequencies but not finite response amplitudes. The incorporation of damping into the calculations, then, allows an approach to finite amplitudes of response under steady-state conditions by equating input energy to dissipated energy. Some systems are not truly uniformly distributed but can be simulated by an equivalent uniformly distributed system. Typical of this class is a plate with equally spaced like stiffeners. The lower numbered modes of this type of stiffened plates will respond in close approximation to mode shapes of uniformly constructed plates.

Some structures have irregular details of mass, stiffness, or end conditions that do not allow a convenient method of solution by distributed system analysis. They must be simulated by a system of lumped masses influenced by local spring effects. The use of these lumped methods is usually limited to a few of the lowest natural frequencies because a lumped system has a finite number of degrees of freedom and the deflection of each mass reflects a single point on the mode shape curve. Good definition of mode shape requires a minimum of about four masses between modes. Practical considerations of the problem solutions limits the number of useable lumped masses. Since design problems are usually associated with low numbered modal response, the lumped-mass method could be of considerable importance.

Any solution of a distributed system dynamic problem is of a complicated nature. A method that allows a simple subjective comprehension of each of the details has considerable value to the analyst. It will increase off-hand evaluation of the effect of any change, and lead to more effective design considerations. The analytic method proposed in this paper takes each specific detail and transforms it into an easily comprehensible part of the system diagram. The method is set up so that it is completely general. One program could be developed to accept all beam problems. Similarly, one program could handle all plate solutions.

The method will be derived by detailed steps so that this paper can serve as a ready reference to users of dynamic analysis.



## SECTION II. METHOD DERIVATION

The first step in the method derivation is to review the basic equations relating moment to deflection (Fig. 1). When the deflection is very small and there is no net

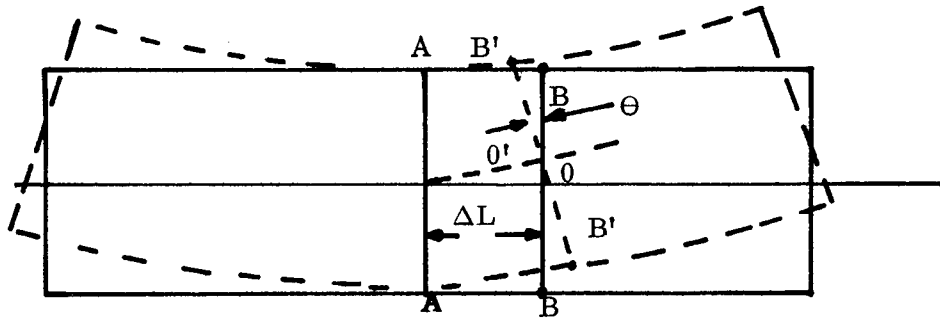


FIGURE 1. SECTION OF A BENT BEAM

elongation of the beam, the deflection of the surface of the beam ( $BB'$ , Fig. 1) can be considered parallel to the undeflected axis of the beam, and the deflection of the neutral axis of the beam ( $00'$ ) may be considered perpendicular to the axis of the beam. If we arbitrarily set  $\Delta L$  equal to  $OB$ , then simple geometric analysis will show that  $BB'$  equals  $00'$  in amplitude, and that they are 90 degrees apart in direction. The equation for the moment acting on section  $B'B'$  is

$$M = EI \frac{\Delta \theta}{\Delta L}, \quad (1)$$

where

$M$  = moment

$E$  = Young's Modulus

$\Delta \theta = \frac{BB'}{\Delta L}$  in radians for this case

$I$  = moment of inertia

$\Delta L$  = unit length of beam.

Now the moment is equal to the product of force,  $F$ , times arm length,  $\Delta L$ . Equation (1) can be written

$$F \Delta L = E I \frac{BB'}{(\Delta L)^2} = E I \frac{00'}{(\Delta L)^2}, \quad (2)$$

or

$$F = \frac{00'}{(\Delta L)^3} E I. \quad (2a)$$

Since the stiffness of the beam,  $K$ , is equal to  $F$  divided by the deflection  $00'$ ,

$$K = \frac{F}{00'} = \frac{EI}{(\Delta L)^3}. \quad (3)$$

Equation (3) has the restrictions that the deflection be small and that moments acting on the stiffness must be acting through the arm length of  $\Delta L$ . Figure I can now be simulated by a descriptive model (Fig. 2).

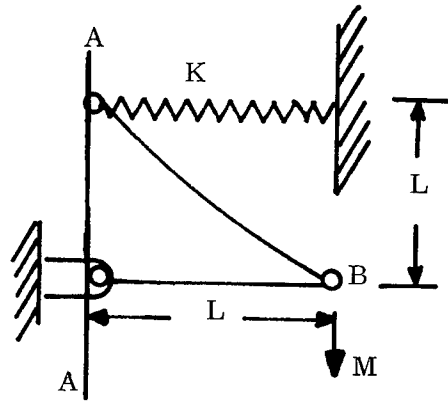
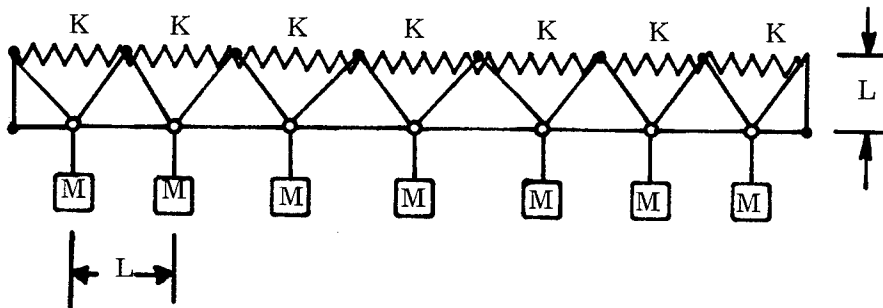


FIGURE 2. BEAM ELEMENT MODEL

To simulate a beam, Figure 2 can be modified to represent a series of linked massless trusses with stiffness and mass lumped arbitrarily and with the basic concepts of Figure 2 adhered to. Such a system is shown in Figure 3. It must be remembered that all effective motion of the spring attachment points is in the horizontal direction, and that of the hinges in the vertical direction (each related to the other). The interrelationship



$K$  = Spring constant  
 $M$  = Mass

FIGURE 3. SIMULATED BEAM

of these two displacements leads to a defining of the spring deflection in terms of the deflection in the vertical axis. Each truss is hinged at the base to the adjacent trusses. As defined above, this hinge allows motion only in the vertical direction. Each hinge deflection is representative of the displacement of the neutral axis of the beam at its relative location. The relative displacement of one hinge of a truss to the other is equal in magnitude to the horizontal displacement of the spring attachment point. Thus all spring deflections can be equated to actual beam deflections. Let us define the net

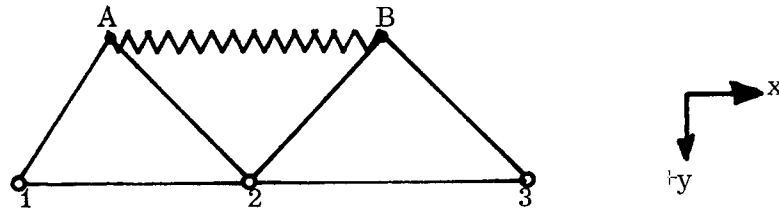


FIGURE 4. SPRING DEFLECTION MODEL

deflection of spring A B, Figure 4, by  $\Delta_{AB}$ , and the displacement of A by  $\Delta_A$ , etc. The following equation develops  $\Delta_{AB}$ :

$$\Delta_A = \Delta_2 - \Delta_1,$$

$$\Delta_B = \Delta_3 - \Delta_2,$$

$$\Delta_{AB} = \Delta_B - \Delta_A = \Delta_3 - 2\Delta_2 + \Delta_1. \quad (4)$$

The spring constants and the masses do not have to be equal for analytic solution. However the solution calculations are greatly simplified when all trusses are of equal size, except for the halves at the ends. This in effect divides the beam into segments of equal length between hinges, with the ends being half segments. Figure 5 depicts the

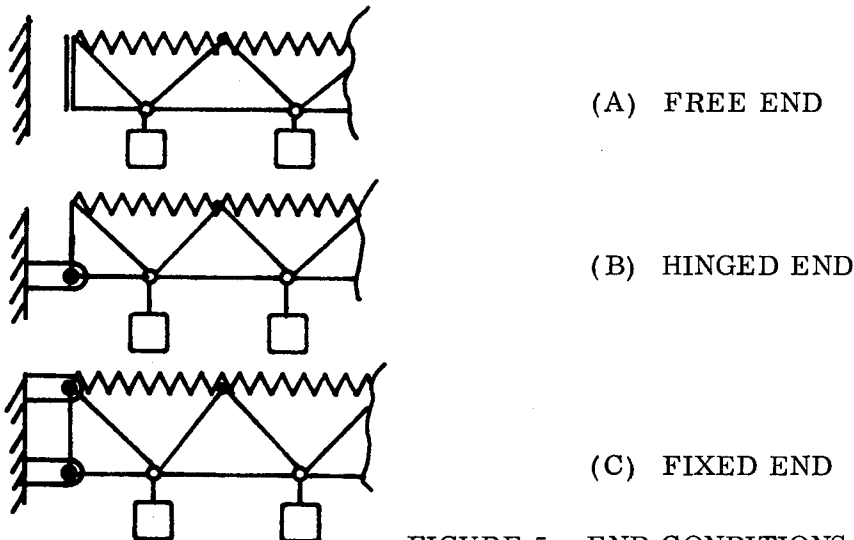
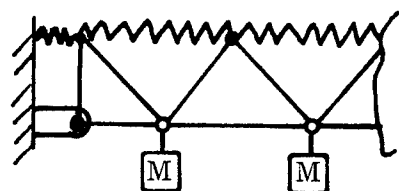
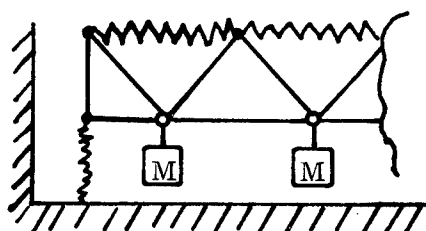


FIGURE 5. END CONDITIONS

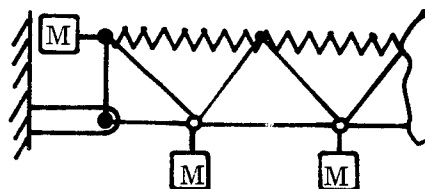
three types of end conditions usually assumed in beam problems. The graphic description is self evident. Now comes some of the real value of this method. Most deviations from the standard end conditions can be simply worked into the solution. Figure 6 shows a few of the possible non-standard end conditions of practical consideration. The non-standard impedance values attached to the ends are easily added to the general equations so that this type of problem is included in the general solution.



(D) RESTRAINED HINGE



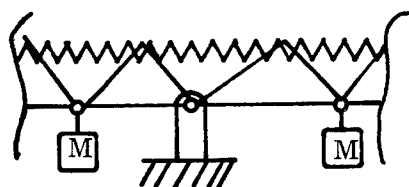
(E) FLEXIBLE ATTACHMENT



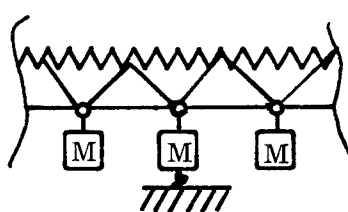
(F) END MASS RESTRAINT

FIGURE 6. NON-STANDARD END CONDITIONS

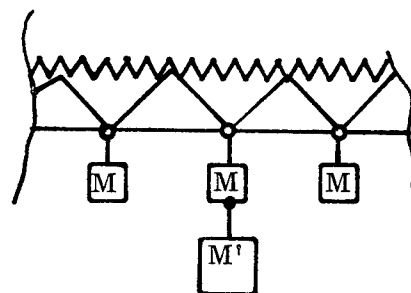
Attachments to the beam also can be accounted for in the general solution. Figure 7 shows the diagrammatic inclusion of such attachments to the simulated system.



(G) FIXED ATTACHMENT



(H) SPRING ATTACHMENT



(I) MASS ATTACHMENT

FIGURE 7. BEAM ATTACHMENTS

It should be noted that damping has not yet been inserted into the diagram. A type of damping simulation is possible but not very practical within the present state-of-the-art. Its usefulness will depend upon further research. The diagrammatic model described up to this point is for free vibration systems. The characteristics of the system can be conveniently represented by impedance values. Such representation can lead to an understanding of the damping properties of the system. Structural damping is inserted into the impedance equation as follows:

$$\begin{aligned}
 Z &= j\omega M + \frac{K(1+j\eta)}{j\omega} \\
 &= j\omega M + \frac{K}{j\omega} + \frac{K\eta}{\omega} .
 \end{aligned}
 \tag{5}$$

$M$  = mass

$K$  = stiffness

$\eta$  = damping loss factor

$\omega$  = angular frequency.

Force,  $F$ , is related to displacement,  $\Delta$ , by the relation

$$F = (Z) (j\omega\Delta) . \tag{6}$$

The simulated system shown in Figure 3 can be solved by using the usual equations of continuity of force and moment. Damping can be included in the impedance equation. However when used, it is recommended to solve the reactive portion first without damping, and then insert damping to calculate amplitudes. For simple beam and plate models, damping can be schematically added as in Figure 8. Any of the usual damping concepts are useable but here we use a damping symbol that is indicative of slipping or Coulombs'

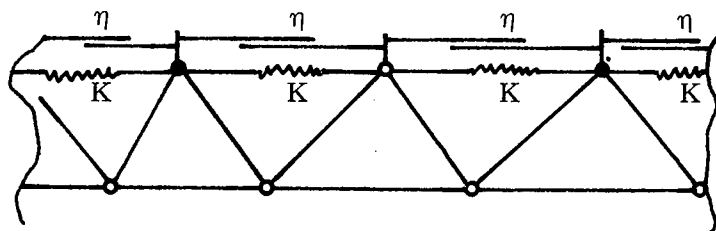


FIGURE 8. SCHEMATIC INCLUSION OF DAMPING

damping and this type is proportionate to displacement. Thus the damping is proportionate to the same displacement that works on the spring located between each of the trusses.

The loss factor is unit length or area dependent. Once the mode shape is determined for a mode the input energy must equal the output energy,

$$\sum_M F_m \Delta_m = \sum_N j\eta K \Delta_n^2 \quad (7)$$

where m is representative of forced input locations and n is representative of response deflections of the various springs. Equation (7) would be applicable to beams or plates without attachments. By including representative attachment loss factor increments, equation (7) could become

$$\sum_M F_m \Delta_m = \sum_N j\eta K \Delta_n^2 + \sum_{N'} j\eta' K \Delta_{n'}^2 \quad (8)$$

where  $N'$ ,  $\eta'$  and  $\Delta_{n'}$  refer to the damping caused by the attachments. The schematic inclusion of the damping caused by the attachments is represented in Figure 9. Equation (8) will solve for discrete values of amplitude for each mode. The  $\Delta$  deflections are dependent on the y displacements. This aspect will be discussed after development of the general solution equation.

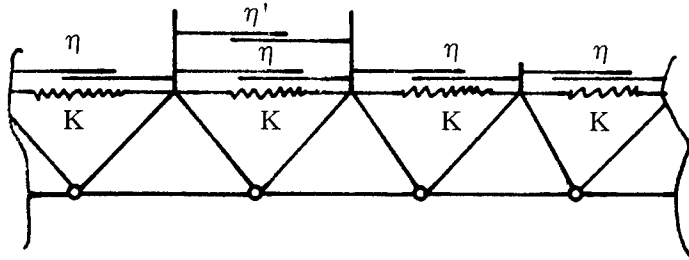


FIGURE 9. INCLUSION OF ATTACHMENT DAMPING

It has been the purpose here to show the possibility of damping analysis by using the analytical model. The remainder of the report will deal with undamped systems only.

### SECTION III. BEAM SOLUTIONS

#### Development of a General Beam Matrix Equation

The beam solution uses a set of equations that define the continuity of force and of moments. Figure 10 is a typical model of a beam of six elements of mass. All motion of the beam is assumed in the  $\pm y$  directions only. End restraints may impose a reaction

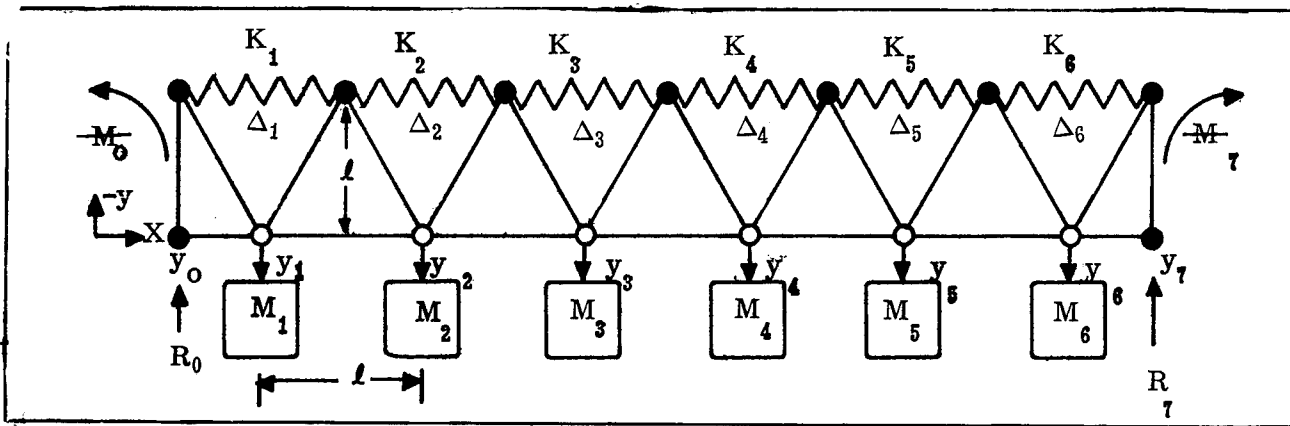


FIGURE 10. MODEL OF A SIX MASS BEAM

FREE

$$\bar{M}_0 = 0, R_0 = 0$$

FIXED

$$y_0 = 0, y_1 = 0$$

HINGED

$$y_0 = 0, \bar{M}_0 = 0$$

FREE

$$\bar{M}_7 = 0, R_7 = 0$$

FIXED

$$y_6 = 0, y_7 = 0$$

HINGED

$$y_7 = 0, \bar{M}_7 = 0$$

$\bar{M}_0, \bar{M}_7$  are end moments, in. lbs.

$R_0, R_7$  are end support forces, lbs.

$K_1$  etc. is stiffness constant =  $\frac{EI}{l^3}$ , lbs./in.

$\Delta_1$  etc. is the net deflection of the spring "K", in.

$y_0$  etc. is the neutral axis deflection of the beam, in.

$l$  is the height and base of each full triangle, in.

$M_1$  etc, is the lumped mass of the section of the beam.

force  $R$  and/or a moment of  $\bar{M}$ . The end may have a displacement of  $y_0$  or  $y_7$ . Each element of mass  $M$  and stiffness  $K$  may have its own assigned quantitative value. In the case of a uniformly distributed system, all  $M$  values would be equal and all  $K$  values would be equal.

The Symbols used in the derivation of the equations are defined below Figure 10. The counter-clockwise moment  $\bar{M}_0$  is considered positive, whereas the clockwise moment  $\bar{M}_7$  is considered positive. This is the usual convention used in beam equations. The deflection of  $y$  in the downward direction is considered positive and the deflections of the springs to the right are considered positive. Both support forces,  $R_0$  and  $R_7$ , are considered positive in the upward direction.

The first set of equations to be evolved are based on the reasoning leading to equation (4). We define  $\Delta_1$  as the deflection imposed on  $K_1$  and  $\Delta_2$  as the deflection imposed upon  $K_2$ , etc., a positive value of  $\Delta$  reflects an elongation of the spring. Remember that the moment arm of the springs is equal in length to the distance between hinges.

$$0 = -\Delta_1 + y_2 - 3y_1 + 2y_0 \quad (9)$$

$$0 = -\Delta_2 + y_3 - 2y_2 + y_1 \quad (10)$$

$$0 = -\Delta_3 + y_4 - 2y_3 + y_2 \quad (11)$$

$$0 = -\Delta_4 + y_5 - 2y_4 + y_3 \quad (12)$$

$$0 = -\Delta_5 + y_6 - 2y_5 + y_4 \quad (13)$$

$$0 = -\Delta_6 + 2y_7 - 3y_6 + y_5 \quad (14)$$

In equations 9 through 14 the y's are actual deflections at the locations as indicated by the subscript.

Next, in Figure 11, let us look at the moments relative to a section 1, 1 of the beam passing through location 1. The deflection in spring  $K_1$  is induced by equal and opposite moments on either side of the section 1, 1. For convenience let us take the moments to the left. Then the continuity of moments at section 1, 1 will be:

$$\Delta_1 K_1 \ell = \cancel{M}_0 - \frac{\ell}{2} R_0.$$

This can be resolved to a more standard form of:

$$0 = \Delta_1 K_1 - \frac{\cancel{M}_0}{\ell} + .5 R_0. \quad (15)$$

Note that  $\cancel{M}_0$  and  $R_0$  result in moments of opposite sign as they resist each other.

Section 2, 2, Figure 12, now involves more than end conditions in the moment equation. Since the springs at locations other than the designated section location only transmit moment, they are not involved in the equation.  $\cancel{M}_0$  and  $R_0$ , as external effects, do not represent physical characteristics of the beam. The mass, on the other hand, is an impedance value and can be treated as such. The moment about section 2, 2 induced by any impedance,  $Z_1$ , with a net displacement of  $y_1$  would be:



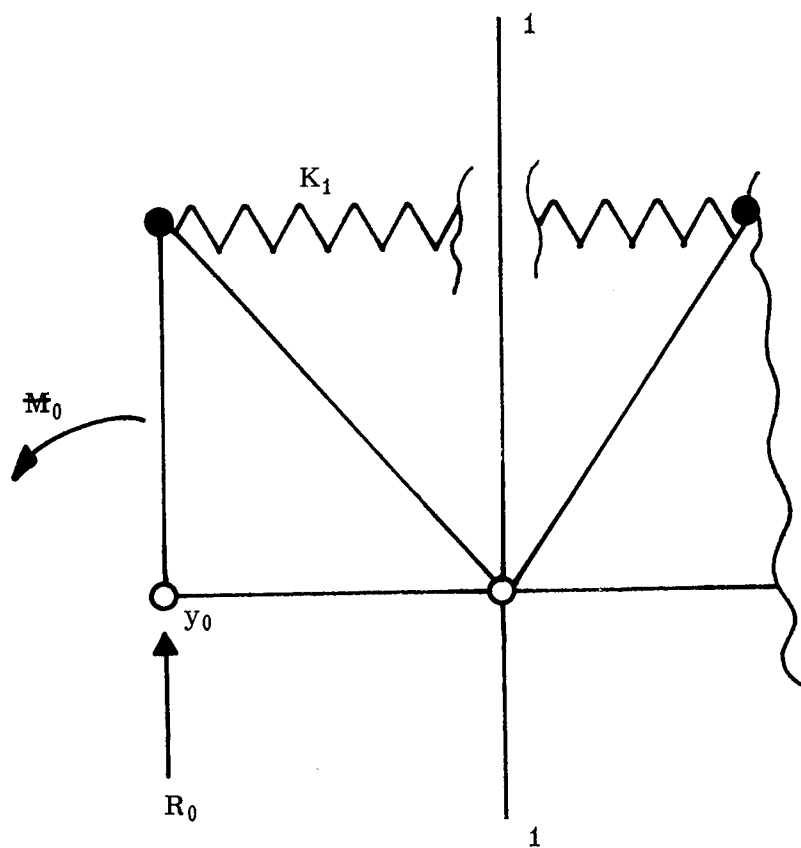


FIGURE 11. BEAM SECTION AT LOCATION 1

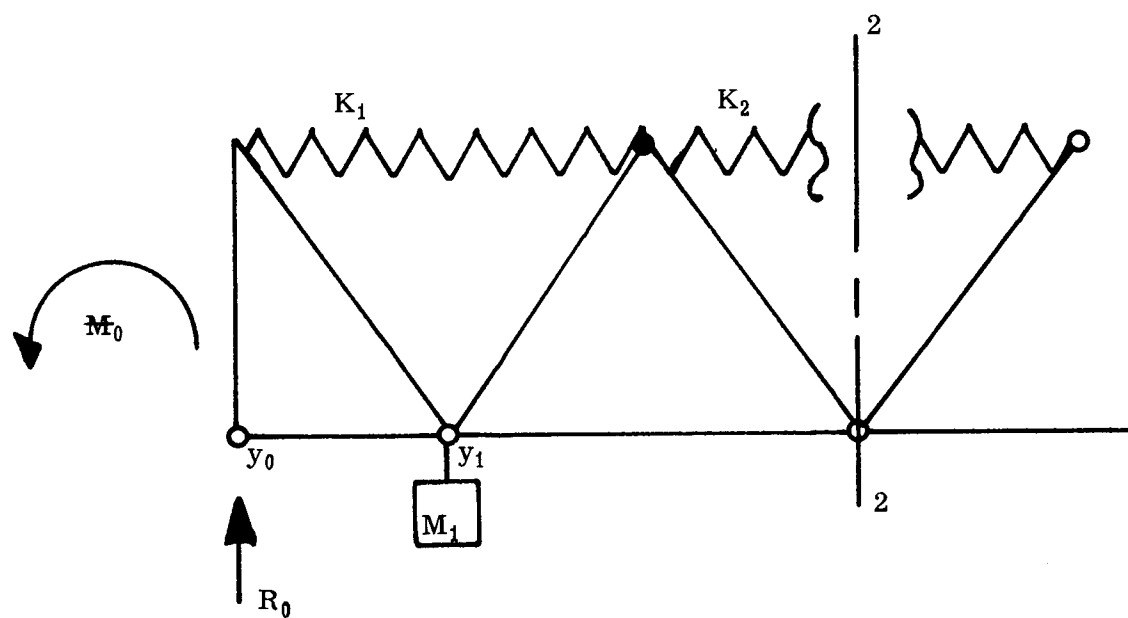


FIGURE 12. BEAM SECTION AT LOCATION 2

$$\bar{M}(Z_1) = (Z_1) (j \omega y_1) \ell.$$

For  $Z_1$  in this case we can substitute  $j \omega M_1$ , then:  $\bar{M}(M_1) = -\omega^2 M_1 y_1 \ell$ . The moments at section 2, 2 can now be equated as:

$$0 = \Delta_2 K_2 - \frac{\bar{M}_0}{\ell} + 1.5 R_0 - \omega^2 y_1 M_1. \quad (16)$$

Following the same reasoning as above, equations of moment continuity can be written for sections at 3, 4, 5, etc. For a six mass beam the set of moment equations would be:

$$0 = \Delta_1 K_1 - \frac{\bar{M}_0}{\ell} + .5 R_0 \quad (15)$$

$$0 = \Delta_2 K_2 - \frac{\bar{M}_0}{\ell} + 1.5 R_0 - \omega^2 y_1 M_1 \quad (16)$$

$$0 = \Delta_3 K_3 - \frac{\bar{M}_0}{\ell} + 2.5 R_0 - 2\omega^2 y_1 M_1 - \omega^2 y_2 M_2 \quad (17)$$

$$0 = \Delta_4 K_4 - \frac{\bar{M}_0}{\ell} + 3.5 R_0 - 3\omega^2 y_1 M_1 - 2\omega^2 y_2 M_2 - \omega^2 y_3 M_3 \quad (18)$$

$$0 = \Delta_5 K_5 - \frac{\bar{M}_0}{\ell} + 4.5 R_0 - 4\omega^2 y_1 M_1 - 3\omega^2 y_2 M_2 - 2\omega^2 y_3 M_3 - \omega^2 y_4 M_4 \quad (19)$$

$$0 = \Delta_6 K_6 - \frac{\bar{M}_0}{\ell} + 5.5 R_0 - 5\omega^2 y_1 M_1 - 4\omega^2 y_2 M_2 - 3\omega^2 y_3 M_3 - 2\omega^2 y_4 M_4 - \omega^2 y_5 M_5 \quad (20)$$

We can also write the moment equation at  $y_7$  and it becomes:

$$0 = -\frac{\bar{M}_7}{\ell} + \frac{\bar{M}_0}{\ell} + 6 R_0 - 5.5 \omega^2 y_1 M_1 - 4.5 \omega^2 y_2 M_2 - 3.5 \omega^2 y_3 M_3 - 2.5 \omega^2 y_4 M_4 - 1.5 \omega^2 y_5 M_5 - 0.5 \omega^2 y_6 M_6. \quad (21)$$

If we substitute equation 9 through 14 into equations 15 through 21 as applicable, we will end with the following list of unknowns:

$$y_0, \bar{M}_0, R_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, \bar{M}_7, R_7.$$

Any choice of end conditions will reduce this group by four, which gives eight unknowns. Thus far we have developed seven equations, numbered 15 through 21. We need one more to satisfy our simultaneous equation solution. In forced vibration the sum of the forces acting in the vertical direction must equal zero. The vertical forces are induced by the end forces  $R_0$  and  $R_7$ , and the inertial forces by the masses or any other restraint at the hinges. The eighth equation can now be developed:

$$0 = -R_0 - R_7 + \omega^2 y_1 M_1 + \omega^2 y_2 M_2 + \omega^2 y_3 M_3 + \omega^2 y_4 M_4 + \omega^2 y_5 M_5 + \omega^2 y_6 M_6 \quad (22)$$

Combining equations 15 through 22 produces matrix equation (23).

0 =	.5	0	$-\ell^{-1}$	0	$+_2K_1$	$-3K_1$	$+K_1$	0	0	0	0	0	0	$R_0$
	1.5	0	$-\ell^{-1}$	0	0	$+(K_2 - \omega^2 M_1)$	$-2K_2$	$+K_2$	0	0	0	0	0	$R_7$
	2.5	0	$-\ell^{-1}$	0	0	$-2\omega^2 M_1$	$+(K_3 - \omega^2 M_2)$	$-2K_3$	$+K_3$	0	0	0	0	$M_0$
	3.5	0	$-\ell^{-1}$	0	0	$-3\omega^2 M_1$	$-2\omega^2 M_2$	$+(K_4 - \omega^2 M_3)$	$-2K_4$	$+K_4$	0	0	0	$M_7$
	4.5	0	$-\ell^{-1}$	0	0	$-4\omega^2 M_1$	$-3\omega^2 M_3$	$-2\omega^2 M_3$	$+(K_5 - \omega^2 M_4)$	$-2K_5$	$+K_5$	0	0	$y_0$
	5.5	0	$-\ell^{-1}$	0	0	$-5\omega^2 M_1$	$-4\omega^2 M_4$	$-3\omega^2 M_3$	$-2\omega^2 M_4$	$+(K_6 - \omega^2 M_5)$	$-3K_6$	$+2K_6$	0	$y_1$
	6	0	$-\ell^{-1}$	$+\ell^{-1}$	0	$-5.5\omega^2 M_1$	$-4.5\omega^2 M_4$	$-3.5\omega^2 M_3$	$-2.5\omega^2 M_4$	$-1.5\omega^2 M_5$	$-.5\omega^2 M_6$	0	0	$y_2$
	-1	-1	0	0	0	$+\omega^2 M_1$	$+\omega^2 M_2$	$+\omega^2 M_3$	$+\omega^2 M_4$	$+\omega^2 M_5$	$+\omega^2 M_6$	0	0	$y_3$
														$y_4$
														$y_5$
														$y_6$
														$y_7$

Matrix Equation, Beam General Model of Six Masses
Equation (23)

Equation (23) represents a completely general mathematical model of a beam. The rectangular matrix and the right-hand column matrix will reduce to solvable square and column matrices of the same order upon selection of the end conditions. Note, from the table below Figure 10, that each end condition has two of the possible variables equal to zero. The selection of end conditions would then reduce the rectangular matrix of equation (23) to a square 8 x 8 matrix and the right-hand column matrix will have four of its variables equal to zero and therefore cancel out to the 8th order.

Since any selection of end conditions will reduce the variables by two, the end conditions need not be alike. The matrix equation will accept any combination of end conditions. This is a distinct advantage of this type of approach. Once the computer is programed to solve the Eigenvalue and Eigenvector problems of full, non-symmetrical, matrices the analytical model will produce solutions not convenient by other methods. Since the matrix is not dependent on uniform constants, a non-symmetrical physical configuration may be solved. In a formal sense of application this method would apply only to stepped structures with uniform constants between steps. In a loose sense the triangular segments may represent a lumping of the variable constants into equivalent values. For instance if the mass of the beam is equatable to a function of position on the beam.

$$M_n = \int_{X_{n-1}}^{X_n} \rho(x) dx$$

where  $M_n$  = Mass of the nth segment of the beam

$\rho$  = Mass density of the beam in mass per unit length

$X$  = distance along the neutral axis of the beam.

The deflection of the beam must be considered linear and estimates made must be for the spring constant of each segment of an irregular beam.

Equation (24) shows the reduction of equation (23) to represent a uniform symmetrical beam of six masses. This specific model is used to illustrate the 3 usual beam conditions of

1. Fixed-Fixed
2. Hinged-Hinged
3. Free-Free

This simplified model, equation (24), was used because it

$$0 = \begin{vmatrix} .5 & -\ell^{-1} & +2K & -3K & +K & 0 \\ 1.5 & -\ell^{-1} & 0 & (K - \omega^2 M) & -2K & +K \\ 2.5 & -\ell^{-1} & 0 & -2\omega^2 M & +(K - \omega^2 M) & -K \\ 3.5 & -\ell^{-1} & 0 & -3\omega^2 M & +(K - 2\omega^2 M) & -(K + \omega^2 M) \end{vmatrix} \begin{vmatrix} R_{0,7} \\ M_{0,7} \\ Y_{0,7} \\ Y_{1,6} \\ Y_{2,5} \\ Y_{3,4} \end{vmatrix}$$

Matrix Equation, Symmetrical Uniform Beam General Model of 6  
Masses

Equation (24)

allowed reasonable (4 by 4) hand solution to show examples of problem solution. Higher order solutions are recommended for the practical use of the mathematical model.

The solution of matrix equations such as equation (23) is somewhat involved. When the rectangular matrix is reduced to a square matrix it represents the impedance of the system. At each resonant frequency the reactive portion of the impedance is equal to zero. Since only reactive impedance has been inserted into the rectangular matrix, the values of  $\omega$  that equate it to zero are solutions of the natural frequencies. These values of  $\omega$  are known as Eigenvalues.

The variables in the column matrix represent system response. Each natural frequency will have a finite set of response values. Solutions of the column matrix are known as Eigenvectors. Even after the insertion of a natural frequency into the square matrix, equation (23) is not solvable. However, if we normalize the response values to one of the Eigenvectors, e.g.,  $y_1$ , then the insertion of unity into the column matrix makes the equation solvable. This means that only relative displacements can be determined from an Eigenvalue - Eigenvector solution. These relative values describe a mode shape for each natural frequency.

The physical reason finite values of response amplitude cannot be had from a solution of equation (25) is that response amplitudes of a steady state forced system are damping dependent. Equation (25) contains no damping values.

Equation (8) equates input energy to dissipated energy. Each deflection  $y$  is equated as

$$y_n = K_n y_y, \quad (25)$$

Where  $y_y$  is the normalization factor. All  $K_n$ 's must be determined. Then the substitution of equation (25) into equation (9) through (14) and the resultants into equation (8) will lead to solutions of finite deflection values for each mode. Usually, at the first few of the lowest natural frequencies such as those suitable for this method of analysis, these deflections will be representative of total deflections at the natural frequencies. This is true because modes other than the resonant mode will have very little amplitude when only low frequencies are being considered.

#### Examples - 4 x 4 Matrix Eigenvalue Solutions of Uniform Beams

The general solution, equation (23), can very often be reduced in scope. This is especially true when the beam is symmetrical and uniform. In fact, equation (24) is the first step in reduction to account for uniformity. When the beam is symmetrical it has an even number of masses and the odd modes only are considered. Each deflection on the right of center should have an equal deflection on the left side. Also the moment and reaction force at the ends are equal. By substitution of one set into the other,

equation (24) can be reduced to half size. Equations 26, 27, and 28 represent uniform symmetrical beams of six masses that reduce to a 4 by 4 system. These 4 by 4 matrices are easily calculated by hand and will be used to check out the accuracy of this analytic method by general comparison with standard solutions.

### Matrix Equations, Symmetrical Uniform Beam, Specific Cases

#### 1. Fixed - Fixed:

$$0 = \begin{vmatrix} .5 & -l^{-1} & +K & 0 \\ 1.5 & -l^{-1} & -2K & +K \\ 2.5 & -l^{-1} & +(K - \omega^2 M) & -K \\ 3.5 & -l^{-1} & +(K - 2\omega^2 M) & -(K + \omega^2 M) \end{vmatrix} \begin{vmatrix} R_{0,7} \\ M_{0,7} \\ y_{2,5} \\ y_{3,4} \end{vmatrix} \quad (26)$$

#### 2. Hinged - Hinged:

$$0 = \begin{vmatrix} .5 & -3K & +K & 0 \\ 1.5 & +(K - \omega^2 M) & -2K & +K \\ 2.5 & -2\omega^2 M & +(K - \omega^2 M) & -K \\ 3.5 & -3\omega^2 M & +(K - 2\omega^2 M) & -(K + \omega^2 M) \end{vmatrix} \begin{vmatrix} R_{0,7} \\ y_{1,6} \\ y_{2,5} \\ y_{3,4} \end{vmatrix} \quad (27)$$

#### 3. Free - Free:

$$0 = \begin{vmatrix} 2K & -3K & +K & 0 \\ 0 & (K - \omega^2 M) & -2K & +K \\ 0 & -2\omega^2 M & +(K - \omega^2 M) & -K \\ 0 & -3\omega^2 M & +(K - 2\omega^2 M) & -(K + \omega^2 M) \end{vmatrix} \begin{vmatrix} y_{0,7} \\ y_{1,6} \\ y_{2,5} \\ y_{3,4} \end{vmatrix} \quad (28)$$

Equation (29) is a longhand solution of a four by four matrix. The calculation will be based upon this solution in tabular form. It should be noted that when A, B, C or D is equal to zero, a portion of the tabulation need not be performed. For example, D equals zero in the solution of the fixed-fixed beam, equation (25).

#### 4 by 4 Matrix Expansion Model

$$\begin{vmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & P & Q \end{vmatrix} = A \begin{vmatrix} F & G & H \\ J & K & L \\ N & P & Q \end{vmatrix} - B \begin{vmatrix} E & G & H \\ I & K & L \\ M & P & Q \end{vmatrix} + C \begin{vmatrix} E & F & H \\ I & J & L \\ M & N & Q \end{vmatrix} - D \begin{vmatrix} E & F & G \\ I & J & K \\ M & N & P \end{vmatrix} \quad (29)$$

$$\begin{aligned} = & \text{AFKQ} + \text{AGLN} + \text{AHJP} - \text{AFPL} - \text{AJGQ} - \text{ANKH} \\ & - \text{BEKQ} - \text{BGLM} - \text{BHJP} + \text{BEPL} + \text{BIGQ} + \text{BMKH} \\ & + \text{CEJQ} + \text{CFLM} + \text{CHIN} - \text{CENL} - \text{CIFQ} - \text{CMJH} \\ & - \text{DEJP} - \text{DFKM} - \text{DGIN} + \text{DENK} + \text{DIFP} + \text{DMJG} \end{aligned}$$

TABLE I. MATRIX EXPANSION OF EQUATION (26)

	$\underline{K^2}$	$\underline{K\omega^2M}$	$\underline{\omega^4M^2}$
$\text{AFKQ} = .5 (-\ell^{-1}) (K - \omega^2M) (-K - \omega^2M) = + .5 \ell^{-1} K^2$			$- .5 \ell^{-1} \omega^4M^2$
$\text{AGLN} = .5 (-2K) (-K) (-\ell^{-1}) = -\ell^{-1} K^2$			
$\text{AHJP} = .5 (K) (-\ell^{-1}) (K - 2\omega^2M) = -.5 \ell^{-1} K^2$		$+ \ell^{-1} \omega^2MK$	
$-\text{AFPL} = -.5 (-\ell^{-1}) (K - 2\omega^2M) (-K) = -.5 \ell^{-1} K^2$		$+ \ell^{-1} \omega^2MK$	
$-\text{AJGQ} = -.5 (-\ell^{-1}) (-2K) (-K - \omega^2M) = + \ell^{-1} K^2$		$+ \ell^{-1} \omega^2MK$	
$-\text{ANKH} = -.5 (-\ell^{-1}) (K - \omega^2M) (K) = + .5 \ell^{-1} K^2$		$- .5 \ell^{-1} \omega^2MK$	
$-\text{BEKQ} = + \ell^{-1} (1.5) (K - \omega^2M) (-K - \omega^2M) = - 1.5 \ell^{-1} K^2$			$+ 1.5 \ell^{-1} \omega^4M^2$
$-\text{BGLM} = + \ell^{-1} (-2K) (-K) (3.5) = + 7 \ell^{-1} K^2$			

Table I (Cont'd)

$-BHIP = +\ell^{-1} (K) (2.5) (K - 2\omega^2 M)$	$= + 2.5\ell^{-1} K^2$	$- 5\ell^{-1} \omega^2 MK$
$+BEPL = -\ell^{-1} (1.5) (K - 2\omega^2 M) (-K)$	$= + 1.5\ell^{-1} K^2$	$- 3\ell^{-1} \omega^2 MK$
$+BIGQ = -\ell^{-1} (2.5) (-2K) (-K - \omega^2 M)$	$= -5\ell^{-1} K^2$	$- 5\ell^{-1} \omega^2 MK$
$+BMKH = -\ell^{-1} (3.5) (K - \omega^2 M) (K)$	$= - 3.5\ell^{-1} K^2$	$+ 3.5\ell^{-1} \omega^2 MK$
$+CEJQ = +K(1.5) (-\ell^{-1}) (-K - \omega^2 M)$	$= + 1.5\ell^{-1} K^2$	$+ 1.5\ell^{-1} \omega^2 MK$
$+CFLM = +K(-\ell^{-1}) (-K) (3.5)$	$= + 3.5\ell^{-1} K^2$	
$+CHIN = +K(K) (2.5) (-\ell^{-1})$	$= - 2.5\ell^{-1} K^2$	
$-CENL = -K(1.5) (-\ell^{-1}) (-K)$	$= - 1.5\ell^{-1} K^2$	
$-CIFQ = -K(2.5) (-\ell^{-1}) (-K - \omega^2 M)$	$= - 2.5\ell^{-1} K^2$	$- 2.5\ell^{-1} \omega^2 MK$
$-CMJH = -K(3.5) (-\ell^{-1}) (K)$	$= + 3.5\ell^{-1} K^2$	

#### 4. Fixed-Fixed Beam Sample Solution

Equation (26) represents the solution of a uniform symmetrical fixed-fixed ended beam. The square matrix equals zero for any natural frequency that can be calculated from the mathematical model. First the matrix has to be expanded to a single equation. By using equation (29) as a model, Table I is constructed. The summation of Table I set to zero is

$$\omega^4 M^2 - 8\omega^2 M K + 3K^2 = 0.$$

If we use the quadratic solution equation of



$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and set

$$a = M^2$$

$$b = -8 M K$$

$$c = 3 K^2,$$

then  $\omega^2$  can be solved as follows:

$$\omega^2 = \frac{+8 M K \pm \sqrt{64 M^2 K^2 - 12 M^2 K^2}}{2 M^2},$$

$$\omega^2 = \frac{+8 M K \pm \sqrt{52 M^2 K^2}}{2 M^2},$$

$$\omega^2 = \frac{+8 M K \pm 7.2111 M K}{2 M^2},$$

$$\omega^2 = \frac{.394 K}{M}, \frac{7.605 K}{M}.$$

Since we have shown  $K = \frac{EI}{\ell^3}$ ,

$$\omega_1^2 = \frac{.394 E I}{M \ell^3}.$$

The standard equation for the first natural frequency of a fixed-fixed beam is

$$\omega_1^2 = \frac{(22.4)^2 E I}{M_t L^3}.$$

In this equation  $M_t = 6 M$  and  $L = 6 \ell$ . These, when substituted in, yield

$$\omega_1^2 = \frac{.386 E I}{M \ell^3}.$$

The error in using the mathematical model of equation (26) would be

$$\text{Error} = \frac{394 - 386}{386} \times 100 = 2.1\%,$$

when  $\omega^2$  is considered. Actually the error in terms of  $\omega$  would be

$$100 \times \frac{\sqrt{394} - \sqrt{386}}{\sqrt{386}} = 100 \times \frac{19.85 - 19.65}{19.65} = \frac{20}{19.65} = 1.04\%.$$

Since the second value of  $\omega^2$  as calculated is about 20 times the first, it is obvious that it does not represent the third mode. We can conclude that the four by four matrix can be good only for the one lowest natural frequency calculation. If an eight by eight matrix were used the accuracy of the first mode would be increased and the third mode made available. Figure 13 is the diagram for equation (26). It is noticeable that only two masses are in oscillation. This is not sufficient freedom to simulate a good third

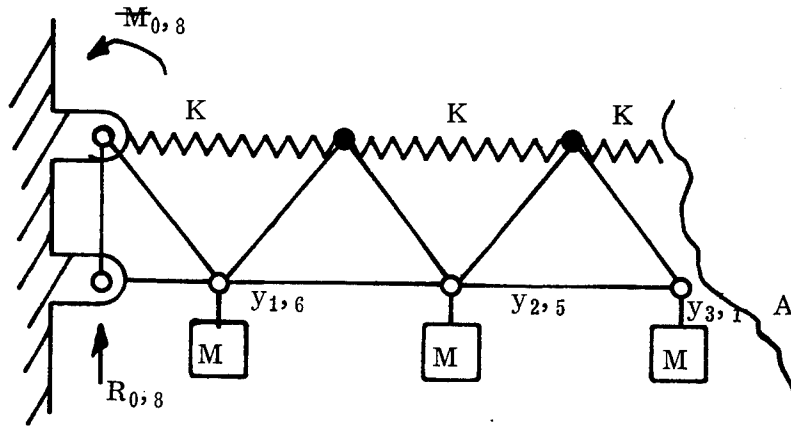


FIGURE 13. FOLDED MODEL FOR FIXED-FIXED BEAM

mode. That is why the third mode can not be calculated by this small size matrix. It should be remembered that the symmetrical method eliminates the second mode.

For the first mode the column matrix of equation (26) can not be determined in absolute values. A solution can be performed in the following manner by using equation (26):

Substitute in the calculated value of  $\omega_1^2$

Divide the column matrix by  $y_3$ .

Solve the set of simultaneous equations for  $R_0/y_3$ ,  $M_0/y_3$  and  $y_2/y_3$ .

Note that  $R_0/y_3$  is the reaction force for the first mode at the beam attachment location per unit deflection of the beam at  $y_3$  and that  $M_0/y_3$  is the moment resulting from the first mode at the beam attachment per unit deflection at  $y_3$ .

Other aspects of the system can be calculated. The average slope of the beam for the first mode between  $y_2$  and  $y_3$  is

$$\left( \frac{y_3}{y_3} - \frac{y_2}{y_3} \right) / \ell,$$

or

$$\left( 1 - \frac{y_2}{y_3} \right) / \ell$$

in radians per unit deflection at  $y_3$ . The maximum stress in the beam is

$$S = \frac{M_0 C}{I}, \quad (30)$$

where  $C$  is the distance from the neutral axis to the extreme fiber. Since we have calculated  $M/y_3$  we get

$$\frac{S}{y_3} = \frac{M_0 C}{y_3 I} \quad (31)$$

as the maximum stress per unit deflection of  $y_3$ . It must be remembered that stress calculated by equation (31) is for one mode of vibration. The total normalized stress resulting from all modes would be the complex summation of

$$\frac{S_t}{y_3} = \sum_{n=1}^{\infty} \frac{S_n}{y} ,$$

where  $S_n/y$  is the normalized maximum stress of the  $n$ th mode.

#### Hinged-Hinged Beam Sample Solution

Equation 27 is a matrix solution for a uniform symmetrical beam with both ends hinged, or simply supported. Equation 27 is solved using equation 29 as a model. Table II is a tabulation of the calculations.

TABLE II. EQUATION (21) MATRIX EXPANSION

AFKQ = .5(K - $\omega^2$ M) (K - $\omega^2$ M) (-K - $\omega^2$ M)	.5 $\omega^6$ M <sup>3</sup>	+.5 $\omega^4$ M <sup>2</sup> K	+.5 $\omega^2$ MK <sup>2</sup>	-.5K <sup>3</sup>
AGLN = .5(-2K) (-K) (-3 $\omega^2$ M)			-3.0	
AHJP = .5(K) (-2 $\omega^2$ M) (K - 2 $\omega^2$ M)	+2.0	-1.0		
-AFPL = -.5(K - $\omega^2$ M) (K - 2 $\omega^2$ M) (-K)	+1.0	-1.5	+5	
-AJGQ = -.5(-2 $\omega^2$ M) (-2K) (-K - $\omega^2$ M)	+2.0	+2.0		
-ANKH = -.5(-3 $\omega^2$ M) (K - $\omega^2$ M) (K)	-1.5	+1.5		
-BEKQ = 3K(1.5) (K - $\omega^2$ M) (-K - $\omega^2$ M)	4.5		-4.5	
-BGLM = 3K(-2K) (-K) (3.5)			+21.0	
-BHIP = 3K(K) (2.5) (K - 2 $\omega^2$ M)		-15.0	+7.5	
+BEPL = -3K(1.5) (K - 2 $\omega^2$ M) (-K)		-9.0	+4.5	
+BIGQ = -3K(2.5) (-2K) (-K - $\omega^2$ M)		-15.	-15.0	
+BMKH = -3K(3.5) (K - $\omega^2$ M) (K)		+10.5	-10.5	
+CEJQ = K(1.5) (-2 $\omega^2$ M) (-K - $\omega^2$ M)	+3.0	+3.0		
+CFLM = K(K - $\omega^2$ M) (-K) (3.5)		+3.5	-3.5	
+CHIN = K(K) (2.5) (-3 $\omega^2$ M)		-7.5		
-CENL = -K(1.5) (-3 $\omega^2$ M) (-K)		-4.5		
-CIFQ = -K(2.5) (K - $\omega^2$ M) (-K - $\omega^2$ M)	-2.5		+2.5	
-CMJH = -K(3.5) (-2 $\omega^2$ M) (K)		+7.0		

The summation of Table II yields:

$$0 = .5\omega^6 M^3 + 9\omega^4 M^2 K - 28.5\omega^2 M K^2 + 2K^3$$

If we set  $\omega^2 = \frac{x K}{M}$  the above equation becomes

$$0 = .5 x^3 + 9 x^2 - 28.5 x + 2.$$

We know that  $x$  will be a small number in the order of 0.1. Therefore the  $x^3$  term in the equation can be neglected. Use the quadratic equation general solution to get

$$x = \frac{28.5 \pm \sqrt{(28.5)^2 - 72}}{18} = .0718,$$

then

$$\omega^2 = \frac{.0718 K}{M} = \frac{.0718 E I}{l^3 M}.$$

The equation for a uniform symmetrical hinged beam is

$$\omega_1^2 = \frac{.0752 E I}{l^3 M}.$$

The error for  $\omega_1^2$  is

$$100 \times \frac{.0752 - .0718}{.0752} = 4.5\%.$$

The error for  $\omega$  is

$$100 \times \frac{\sqrt{752} - \sqrt{718}}{\sqrt{752}} = 2.3\%.$$

As stated in the fixed-fixed beam example, the 4 by 4 matrix is in reality too small for good problem solution. The third mode solution is not available. The normalized moment, end resultant force and deflections for the first mode can be calculated as well as the average slope between hinges.

TABLE III. MATRIX EXPANSION OF EQUATION (28)

$2 K(K - \omega^2 M) (K - \omega^2 M) (-K - \omega^2 M)$	=	$-2K^4 + 2K^3 \omega^2 M + 2K^2 \omega^4 M^2 - 2K \omega^6 M^3$
$2 K(-2K) (-K) (-3\omega^2 M)$	=	$-12 K^3 \omega^2 M$
$2 K(-2\omega^2 M) (K - 2\omega^2 M) (+K)$	=	$-4K^3 \omega^2 M + 8 K^2 \omega^4 M^2$

Table III (Cont'd)

$$\begin{array}{rcl}
 -2 K(-3\omega^2 M) (K - \omega^2 M) (K) & = & 6 K^3 \omega^2 M - 6 K^2 \omega^4 M^2 \\
 -2 K(K - \omega^2 M) (K - 2\omega^2 M) (-K) & = & 2 K^4 - 6 K^3 \omega^2 M + 4 K^2 \omega^4 M^2 \\
 -2 K(-2 \omega^2 M) (-2 K) (-K - \omega^2 M) & = & + 8 K^3 \omega^2 M + 8 K^2 \omega^4 M^2 \\
 & & \hline
 & & - 6 K^3 \omega^2 M + 16 K^2 \omega^4 M^2 - 2 K \omega^6 M^3
 \end{array}$$

### Free-Free Beam Sample Solution

Equation (28) is the matrix solution for a uniform symmetrical beam with both ends free. The square matrix of equation (28) reduces to:

$$2 K \begin{vmatrix} K - \omega^2 M & - 2 K & + K \\ - 2\omega^2 M & + K - \omega^2 M & - K \\ - 3 \omega^2 M & + K - 2\omega^2 M & - K - \omega^2 M \end{vmatrix} .$$

The solution of this matrix when set equal to zero is from Table III and is

$$0 = - 2 \omega^6 M^3 K + 16 \omega^4 M^2 K^2 - 6 \omega^2 M K^3 .$$

If we let  $\omega^2 = \frac{x K}{M}$ , then the above equation reduces to

$$0 = + 2 x^2 - 16 x + 6 = .395 .$$

Then,

$$\omega_1^2 = \frac{.395 K}{M} = \frac{.395 E I}{\ell^3 M} .$$

The first natural frequency of the free-free beam should be

$$\omega_1^2 = \frac{.386 E I}{M \ell^3} .$$

The error in terms of  $\omega_1^2$  will be

$$100 \times \frac{.395 - .386}{.395} = 2.3\%.$$

The error in terms of  $\omega$  will be

$$100 \times \frac{\sqrt{395} \sqrt{386}}{\sqrt{396}} = 1.1\%.$$

### Cantilever Beam Sample Solution

The cantilever beam is a special case of the beam equation. It has one end fixed and one end free. Thus it is not a symmetrical beam. Figure 14 represents a diagrammatic model that fits a 4 by 4 matrix. This particular model is used because it utilizes the same model configurations of equations 26 through 28. Moments are taken at the hinge locations. Equation (32) represents the summation of moments about the first mass,  $M_1$ , by using the left input to balance the moment of spring  $K_1$ . Equation (33) lists the summation of moments that are to the left of mass  $M_2$  and reacting on spring  $K_2$ . Equation (34) represents the summation of the moments acting to the right of mass  $M_2$  and acting on spring  $K_2$ . This switching of sides is convenient because spring  $K_3$ , according to this model, has no deflection. Equation (35) is the summation of forces acting in the vertical direction. Equations 36 and 37 equate the spring deflections to hinge displacements. Equation 38 follows from equations (32) through (38). Table IV is the solution of the square matrix of equation (38).

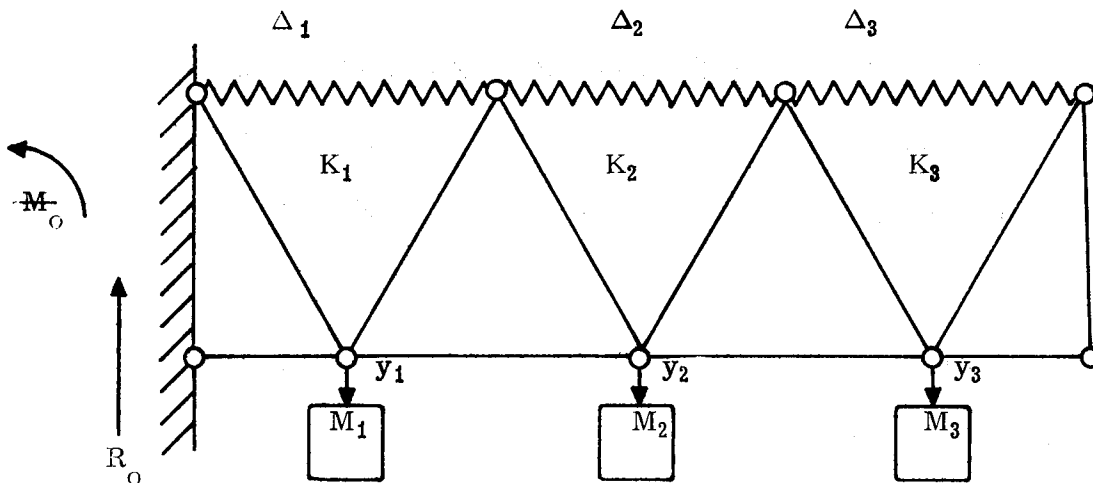


FIGURE 14. MODEL OF A CANTILEVER BEAM

# Cantilever equations

$$0 = -\Delta_1 K_1 + \frac{M_0}{\ell} - .5 R_0 \quad (32)$$

$$0 = -\Delta_2 K_2 + \frac{M_0}{\ell} - 1.5 R_0 \quad (33)$$

$$0 = -\Delta_2 K_2 + \omega^2 M_3 y_3 \quad (34)$$

$$0 = -R_0 + \omega^2 M_2 y_2 + \omega^2 M_3 y_3 \quad (35)$$

$$\Delta_1 = y_2 \quad (36)$$

$$\Delta_2 = y_3 - 2y_2 \quad (37)$$

$$0 = \begin{vmatrix} \ell^{-1} & -.5 & -K & 0 \\ \ell^{-1} & -1.5 & +2K & -K \\ 0 & 0 & +2K & -K + \omega^2 M \\ 0 & -1 & \omega^2 M & \omega^2 M \end{vmatrix} \begin{vmatrix} M_0 \\ R_0 \\ y_2 \\ y_3 \end{vmatrix} \quad (38)$$

TABLE IV. MATRIX EXPANSION OF EQUATION (38)

$\ell^{-1} (-1.5) (2K) (\omega^2 M)$	=	$-3 \omega^2 K M \ell^{-1}$
$\ell^{-1} (2K) (-1) (-K + \omega^2 M)$	=	$2K^2 \ell^{-1} - 2 \omega^2 K M \ell^{-1}$
$\ell^{-1} (0)$	=	0
$-\ell^{-1} (-1) (2K) (-K)$	=	$-2 K^2 \ell^{-1}$
$-\ell^{-1} (-1.5) (\omega^2 M) (-K + \omega^2 M)$	=	$-1.5 \omega^2 K M \ell^{-1} + 1.5 \omega^4 M^2 \ell^{-1}$
$-\ell^{-1} (0)$	=	0
$.5 (\ell^{-1}) (+2 K) (\omega^2 M)$	=	$+ \omega^2 K M \ell^{-1}$
$.5 (0)$	=	0



Table IV (Cont'd)

.5 (0)	=	0	
-.5 (0)	=	0	
-.5 (0)	=	0	
$-.5 \ell^{-1}(\omega^2 M) (-K + \omega^2 M)$	=	$+0.5 \omega^2 K M \ell^{-1}$	$-0.5 \omega^4 M^2 \ell^{-1}$
-K (0)	=	0	
-K (0)	=	0	
-K (0)	=	0	
+K (0)	=	0	
K (0)	=	0	
$K \ell^{-1} (-1) (-K + \omega^2 M)$	=	$K^2 \ell^{-1}$	$-\omega^2 K M \ell^{-1}$
Total of Table IV		$K^2 \ell^{-1}$	$-6 \omega^2 K M \ell^{-1} + \omega^4 M^2 \ell^{-1}$

Setting the square matrix equal to zero we get

$$0 = \omega^4 M^2 - 6 \omega^2 K M + K^2.$$

Let

$$\omega^2 = \frac{x K}{M},$$

then

$$0 = x^2 - 6 x + 1,$$

$$x = \frac{6 \pm \sqrt{36 - 4}}{2} = \frac{6 \pm \sqrt{32}}{2},$$

$$x = \frac{6 - 5.67}{2} = \frac{.33}{2} = .165,$$

then

$$\omega_1^2 = \frac{.165 K}{M} = \frac{.165 E I}{\ell^3 M} \quad .$$

The standard solution for the first mode of a cantilever beam will give

$$\omega_1^2 = \frac{.153 E I}{M \ell^3} \quad .$$

The error in terms of  $\omega^2$  is

$$\frac{.165 - .153}{.153} \times 100 = 7.9\% \quad .$$

The error in terms of  $\omega$  is

$$\frac{\sqrt{163} - \sqrt{153}}{\sqrt{153}} \times 100 = 3.2\% \quad .$$

Since there are only two displacement values in this example,  $y_2$  and  $y_3$ , good accuracy can not be expected.

### Summary of the Worked Examples

The four examples worked here are algebraic with the end resultant in a form that could be compared to the general beam natural frequency equation. For an example where the masses, stiffnesses, and length dimensions can be determined, the Eigenvalue solution of the square matrix will find the natural frequencies directly without the substitution of  $\frac{E I}{\ell^3}$  in for K at the end of the problem. Actually this substitution will be made when determining the initial values of stiffness to insert into the matrix. It need only be made once.

Although a 4 by 4 matrix is obviously too small to assure enough input detail, the accuracy of the four problems is good enough to point out the validity of the method.

### Non-Uniform Beam Solutions

The mathematical model put forth in this paper is adaptable to beams of non-uniform stiffness and/or mass as well as beams with a curved neutral axis. These problems can be developed to fit a standard Eigenvalue solution to a square matrix but the

matrix itself is, of course, not standard in form. Each of these special situations must have a special set of equations developed. First we must examine the limitations of the method. For an example, consider Figure 15 as typical of a special beam base.

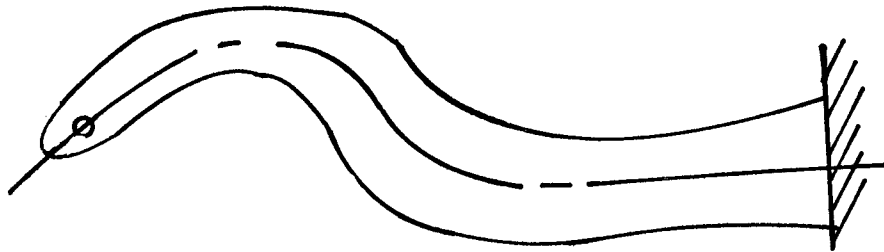


FIGURE 15. NON-UNIFORM, NON SYMMETRICAL BEAM

Each link of the simulated model, Figure 16, must have like dimensions but the individual values of  $M$  and  $K$  can vary. The trussed links must be small enough so that the net deflection of the beam hinge can be considered a net deflection of the spring. Error will be proportionate to the relative angle made by the bases of adjacent trusses. This angle will increase as the length " $l$ " is increased, and decrease as the length " $l$ " is decreased. The allowable error puts a limitation on the model truss size and therefore on the minimum number of trusses.

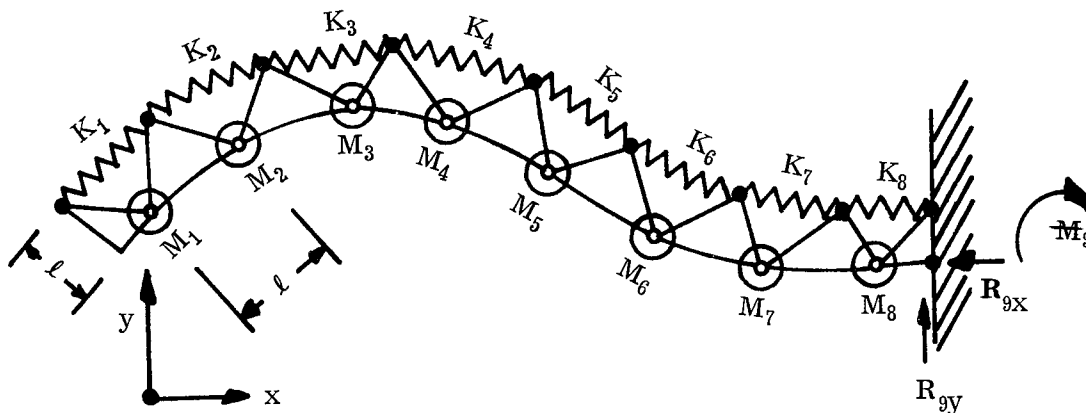


FIGURE 16. NON-UNIFORM BEAM MODEL

The deflection at  $M_1$  will have a resultant with both an " $x$ " and a " $y$ " component. However, if a single coordinate system of response relative to location is worked out, a smaller matrix results.

In this particular model there is no moment on spring  $K_1$ . The motion of each mass concentrated at the hinge is considered to be in the direction of a ray that bisects the angle between the adjacent truss bases. The moment arm about  $M_1$  will then be a perpendicular struck through the location of  $M_1$  to the directional ray. By using this method a set of moment equations can be developed in the same manner as for the straight beam. The force summation must equal zero in both the x and the y direction. A set of equations can be developed to set up a matrix equation that will solve for Eigenvalues and Eigenvectors.

The use of the mathematical model presented herein probably has its greatest value in helping to solve the more complicated situations by reducing the specifics to a model that can be solved.

#### SECTION IV. VARIATIONS ON THE ANALYTICAL BEAM MODEL

##### Initial Displacement

If an initial displacement is known, it replaces the appropriate  $y$ . For example: if  $y_4$  has an initial displacement of .005 inches, this value will replace  $y_4$  in the column matrix of equation (23).

##### Mode Shape

The maximum displacement is given an initial displacement of unity. A plot of the values of  $y$  versus location on the beam will describe the mode shape. Curve fitting procedure will equate  $y$  to  $x$ .

##### Forced Oscillation

Forced oscillations will enter the matrix equation (23) as a column matrix on the left side. The procedure is illustrated by Figure 17.

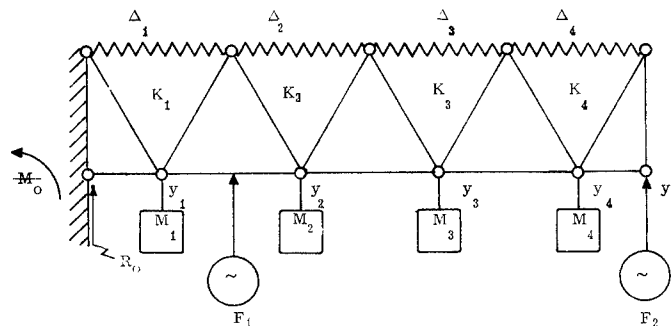


FIGURE 17. FORCED EXCITATION EXAMPLE

$$\begin{aligned}
0 &= -\Delta_1 K_1 + \frac{\bar{M}_0}{\ell} - .5 R_0 \\
+ .5 F_1 &= -\Delta_2 K_2 + \frac{\bar{M}_0}{\ell} - 1.5 R_0 \\
+ 1.5 F_1 &= -\Delta_3 K_3 + \frac{\bar{M}_0}{\ell} - 2.5 R_0 + \omega^2 M_2 y_2 \\
+ 2.5 F_1 &= -\Delta_4 K_4 + \frac{\bar{M}_0}{\ell} - 3.5 R_0 + 2\omega^2 M_2 y_2 + \omega^2 M_3 y_3 \\
+ .5 F_2 &= -\Delta_4 K_4 \\
+ (F_1 + F_2) &= -R_0 + \omega^2 M_2 y_2 + \omega^2 M_3 y_3 + \omega^2 M_4 y_4
\end{aligned}$$

$$\Delta_1 = y_2$$

$$\Delta_2 = y_3 - 2y_2$$

$$\Delta_3 = y_4 - 2y_3 + y_2$$

$$\Delta_4 = 2y_5 - 3y_4 + y_3$$

$$\begin{array}{c|c|c|c|c|c|c|c|c}
0 & & - .5 & +\ell^{-1} & -K_1 & 0 & 0 & 0 & R_0 \\
+ .5 F_1 & & -1.5 & +\ell^{-1} & +2K_2 & -K_2 & 0 & 0 & \bar{M}_0 \\
+ 1.5 F_1 & & -2.5 & +\ell^{-1} & (-K_3 + \omega^2 M_2) & +2K_3 & -K_3 & 0 & y_2 \\
+ 2.5 F_1 & = & -3.5 & +\ell^{-1} & + 2\omega^2 M & (-K_4 + \omega^2 M) & + 3K_4 & -2K_4 & y_3 \\
+ .5 F_2 & & 0 & 0 & 0 & -K_4 & +3K_4 & -2K_4 & y_4 \\
+ (F_1 + F_2) & & -1 & 0 & \omega^2 M_2 & \omega^2 M_3 & \omega^2 M_4 & 0 & y_5
\end{array}$$

### Mixed End Conditions

When the end of the beam is anchored to a flexible member,  $\ell \Delta_0 K_0$  is substituted in equation (23) for  $\bar{M}_0$  and  $\Delta_0 = 2y_1$ ;  $y_1$  is then a variable and replaces  $\bar{M}_0$  in the right side column matrix.

When the end condition has a rotational oscillation,  $f(x_0)$  is then a known input  $f(x_0) = 2y_1$  and its function is substituted into the right hand column matrix for  $y_1$ .

When the end is oscillating in the  $y$  direction,  $f(y_0)$  is a known function and is substituted into the right hand column matrix (Fig. 18).

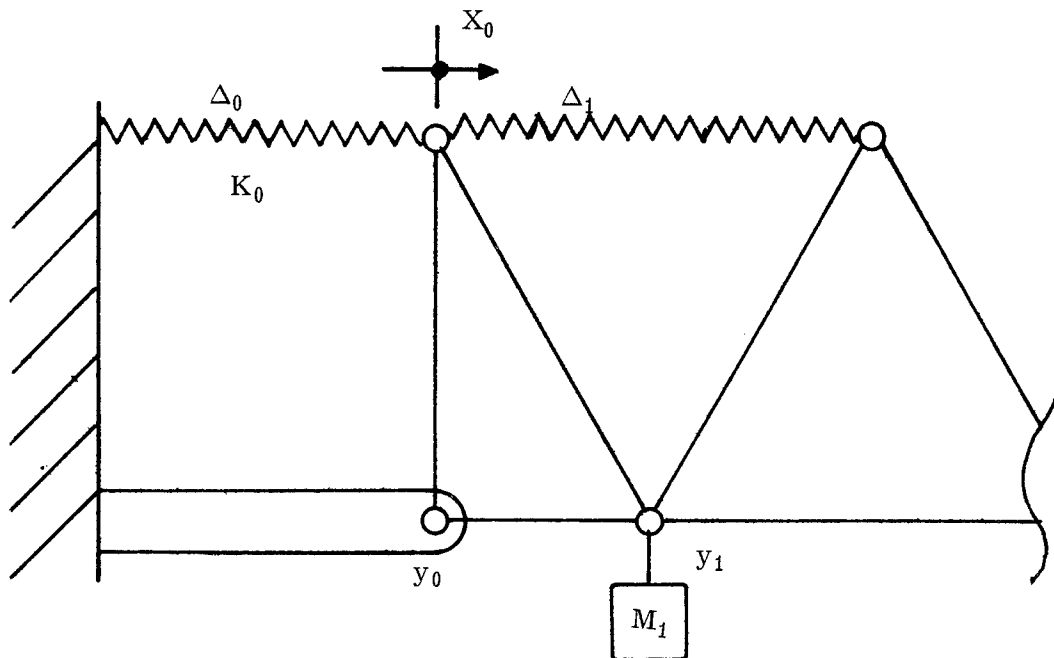


FIGURE 18. EXAMPLE OF FLEXIBLE END ATTACHMENT

### Multi-Supported Beams

For a multi supported beam,  $R_n$  must eliminate  $y_n$  in the matrix equation. Thus  $R_n$  will replace  $\omega^2 M_n y_n$  in the basic equations and  $R_n$  will replace  $y_n$  in the right hand column matrix as an unknown (Fig. 19).

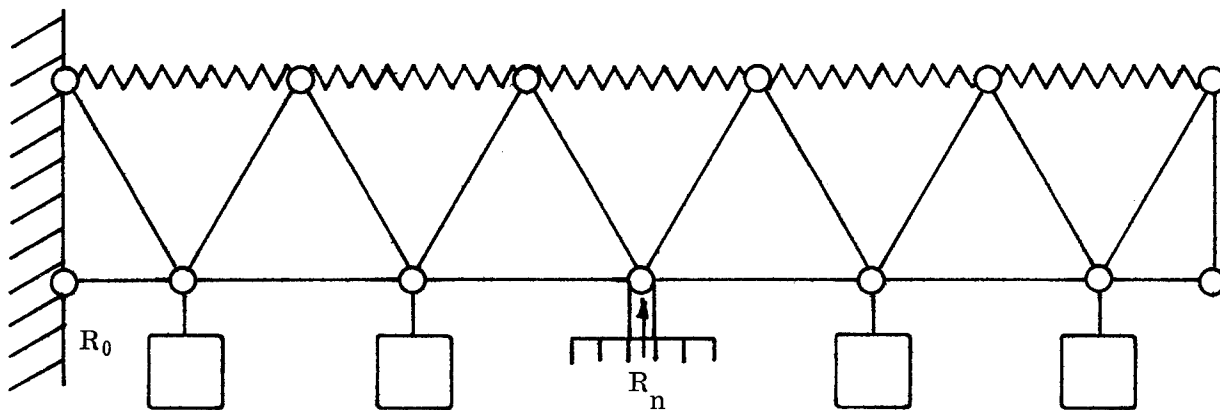


FIGURE 19. EXAMPLE OF MULTI SUPPORTED BEAM

In the non-rigid supported beam, as illustrated, the support must be at a hinge between triangles. In each instance of support, for example where  $\omega^2 M_1 y_1$  and  $\omega^2 M_5 y_5$  are in the matrix (23), the quantities  $(\omega^2 M_1 - K'_1)$  and  $(\omega^2 M_5 - K'_5)$  are respectively substituted since the spring will resist motion at the support. If the springs are initially not at rest, the respective forces must be inserted (Fig. 20).

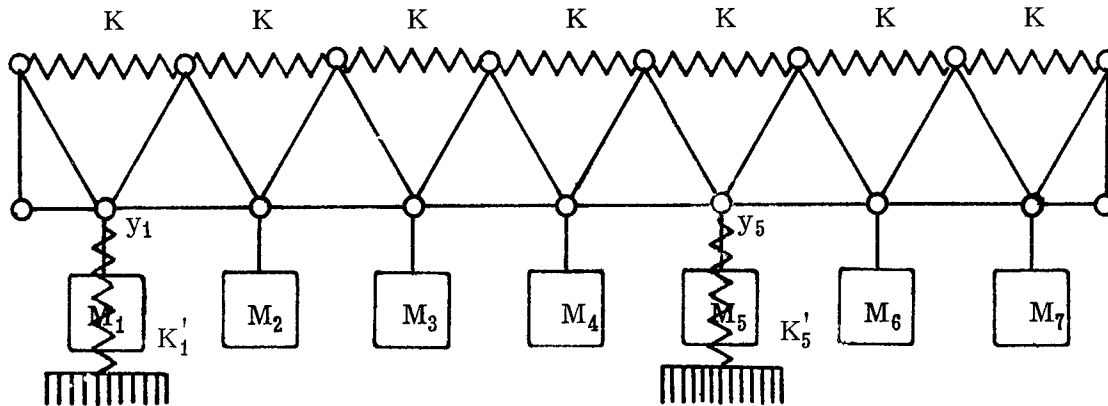


FIGURE 20. EXAMPLE OF FLEXIBLE SUPPORTED BEAM

#### SECTION V. ADAPTION OF A BEAM ANALYTICAL MODEL TO A PLATE ANALYTICAL MODEL

The beam of Figure 3 is simulated by a model that is most convenient for all of the beam problems. Other models may be constructed to meet specific purposes. The beam of Figure 3 is not readily adaptable to plate configurations. Therefore, it is necessary to modify our approach and construct a basically new model as shown in Figure 21. This new model represents a change in mass attachment that complicates the coupling of mass to the column matrix of variables.

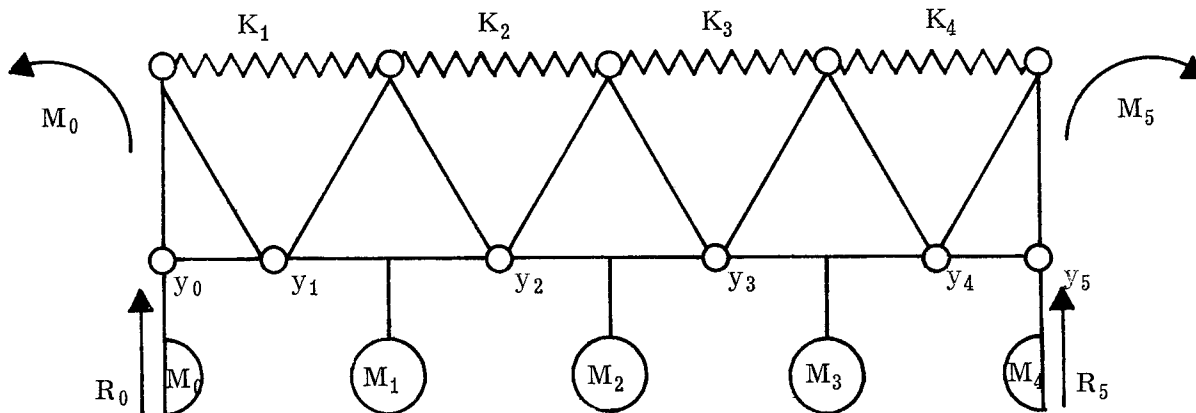


FIGURE 21. MODIFIED ANALYTICAL MODEL

The displacements of  $M_1, M_2$  and  $M_3$  are the average of the two adjacent hinge displacements. Thus the dynamic force of  $M_1$  may be represented by  $\omega^2 M_1 \left( \frac{y_1 + y_2}{2} \right)$ .

In order to expand the beam to a two dimensional model, the triangle concept is replaced with a massless rectangular pyramid as shown in Figure 22 and Figure 23.

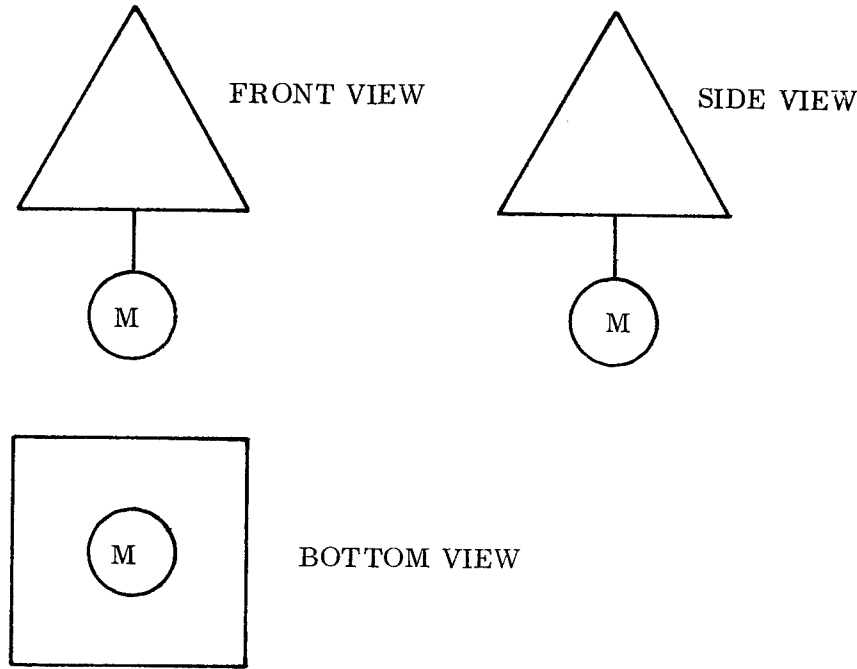


FIGURE 22. PYRAMID CONFIGURATION

In the development of the matrix equation for this beam the mass displacement may be taken as average of two diagonal adjacent hinges; for convenience, the top left and bottom right of each pyramid base are taken. The displacement of spring  $K_{a1}$  is

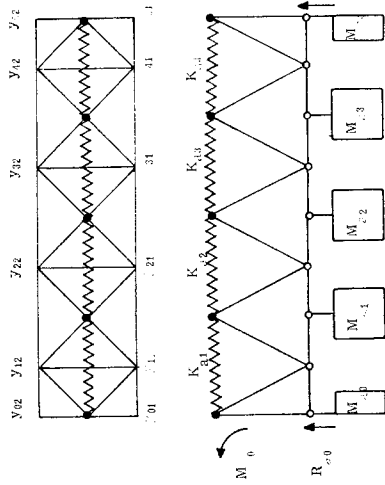
$$\Delta_{a1} = \left( \frac{y_{21} + y_{22}}{2} \right) - 3 \left( \frac{y_{12} + y_{11}}{2} \right) + 2 \left( \frac{y_{01} + y_{02}}{2} \right)$$

and the displacement of  $K_{a2}$  is

$$\Delta_{a2} = \left( \frac{y_{32} + y_{31}}{2} \right) - 2 \left( \frac{y_{22} + y_{21}}{2} \right) + \left( \frac{y_{12} + y_{11}}{2} \right)$$

The first six lines of matrix equation (39) is the analytical model of the two-dimensional, four-spring beam. This model has no advantages for beam problems over equation 23 and is more complicated; but the corner hinges allow for modal coupling of a plate problem.





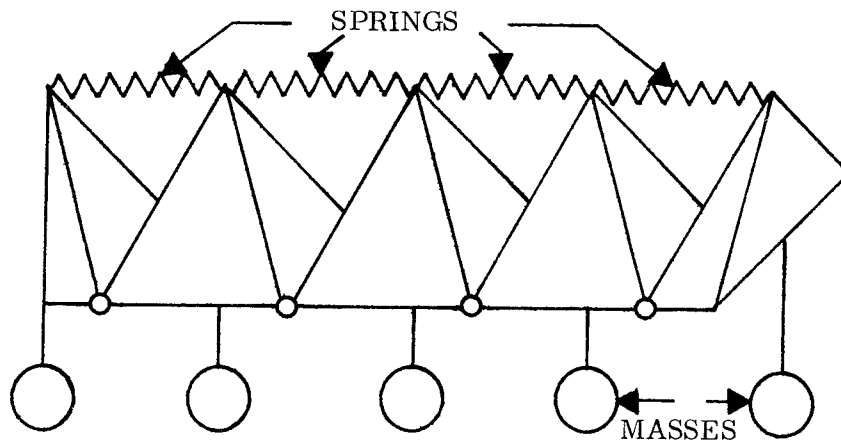
Four Mass Beam Modified Model

$$\begin{array}{c}
 \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] = \left[ \begin{array}{c} +\frac{\omega^2}{2} M_{a0} \\ -(K_{a1} - \frac{\omega^2}{4} M_{a0}) + \frac{3}{2} K_{a1} - \frac{1}{2} K_{a1} \\ +\frac{3}{4} \omega^2 M_{a0} - \frac{1}{2} K_{a2} + (K_{a2} + \frac{\omega^2}{4} M_{a1}) - \frac{1}{2} K_{a2} \\ 0 - \frac{1}{2} K_{a3} + \frac{3}{4} \omega^2 M_{a1} + (+K_{a3} + \frac{\omega^2}{4} M_{a2}) - \frac{1}{2} K_{a3} \\ +\frac{3}{4} \omega^2 M_{a0} 0 + \frac{5}{4} \omega^2 M_{a1} - (\frac{1}{2} K_{a4} - \frac{3}{4} \omega^2 M_{a2}) + \frac{3}{2} K_{a4} + \frac{1}{4} \omega^2 M_{a3} \\ 0 + 2 \omega^2 M_{a0} 0 + \frac{3}{2} M_{a1} + \omega^2 M_{a2} + \frac{1}{2} \omega^2 M_{a3} \end{array} \right] \left[ \begin{array}{c} +\frac{\omega^2}{2} M_{a4} \\ +\frac{\omega^2}{2} M_{a0} \\ +\frac{\omega^2}{4} M_{a0} - (K_{a1} - \frac{\omega^2}{4} M_{a0}) + \frac{3}{2} K_{a1} - \frac{1}{2} K_{a1} \\ 0 0 + \frac{3}{4} \omega^2 M_{a0} + \frac{5}{4} \omega^2 M_{a1} + \frac{7}{4} \omega^2 M_{a0} - K_{a4} \\ 0 0 + 2 \omega^2 M_{a0} \end{array} \right] \left[ \begin{array}{c} +\frac{\omega^2}{2} M_{a1} \\ +\frac{\omega^2}{2} M_{a2} \\ +\frac{\omega^2}{2} M_{a3} \\ +\frac{\omega^2}{2} M_{a4} \\ +\frac{\omega^2}{2} M_{a0} \\ +\frac{\omega^2}{2} M_{a1} \\ +\frac{\omega^2}{2} M_{a2} \\ +\frac{\omega^2}{2} M_{a3} \\ +\frac{\omega^2}{2} M_{a4} \end{array} \right] \left[ \begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} y_{01} \\ y_{11} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{51} \\ y_{12} \\ y_{22} \\ y_{32} \\ y_{42} \\ y_{52} \\ R_{a0} \\ R_{a5} \\ M_{a0} \\ M_{a5} \end{array} \right]
 \end{array}$$

When one equation is to be omitted, line (6) is deleted and line (7) is substituted for line (5)

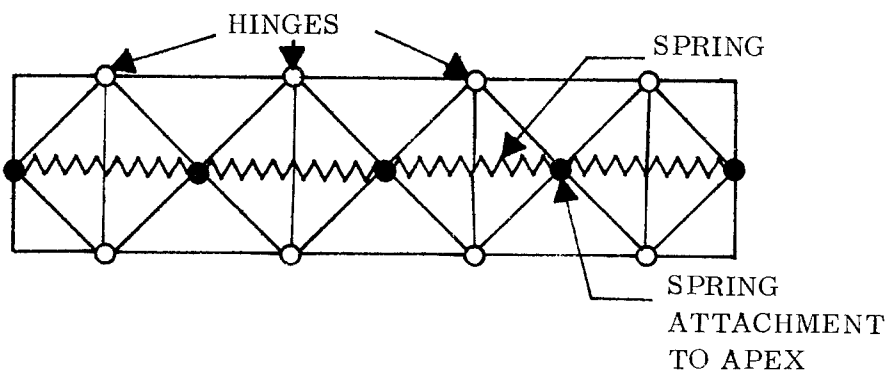
$$0 = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} +\frac{\omega^2}{2} M_{a4} \\ +\frac{\omega^2}{2} M_{a0} \\ +\frac{\omega^2}{4} M_{a0} - (K_{a1} - \frac{\omega^2}{4} M_{a0}) + \frac{3}{2} K_{a1} - \frac{1}{2} K_{a1} \\ 0 0 + \frac{3}{4} \omega^2 M_{a0} + \frac{5}{4} \omega^2 M_{a1} + \frac{7}{4} \omega^2 M_{a0} - K_{a4} \\ 0 0 + 2 \omega^2 M_{a0} \end{array} \right] \left[ \begin{array}{c} +\frac{\omega^2}{2} M_{a1} \\ +\frac{\omega^2}{2} M_{a2} \\ +\frac{\omega^2}{2} M_{a3} \\ +\frac{\omega^2}{2} M_{a4} \\ +\frac{\omega^2}{2} M_{a0} \\ +\frac{\omega^2}{2} M_{a1} \\ +\frac{\omega^2}{2} M_{a2} \\ +\frac{\omega^2}{2} M_{a3} \\ +\frac{\omega^2}{2} M_{a4} \end{array} \right] \left[ \begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} y_{01} \\ y_{11} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{51} \\ y_{12} \\ y_{22} \\ y_{32} \\ y_{42} \\ y_{52} \\ R_{a0} \\ R_{a5} \\ M_{a0} \\ M_{a5} \end{array} \right]$$

These pyramids are linked at the corners to form a beam as in Figure 23

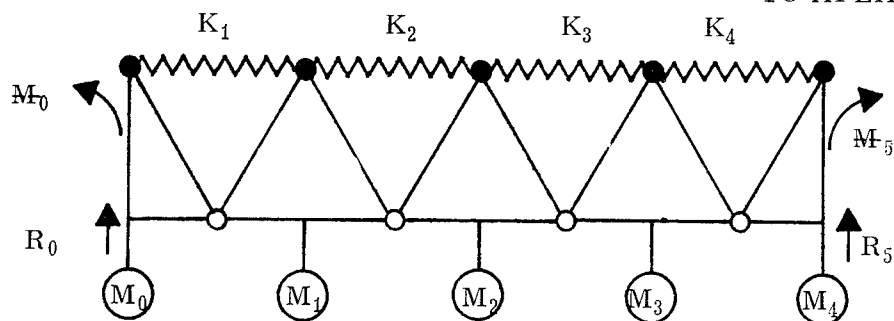


GENERAL VIEW

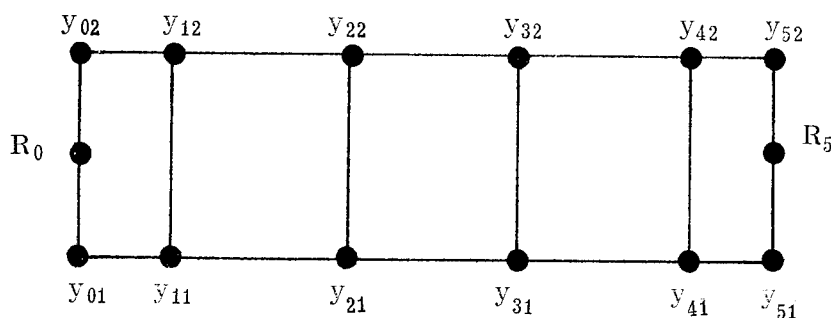
4-SPRING BEAM  
2-DIMENSIONAL  
CONCEPT



TOP VIEW



SIDE VIEW FOR  
NOMENCLATURE



TOP VIEW FOR  
NOMENCLATURE

FIGURE 23. PYRAMIDIC MODEL SUITABLE FOR PLATE ADAPTION

The plate made up of four-spring beams may be represented as in Figure 24.

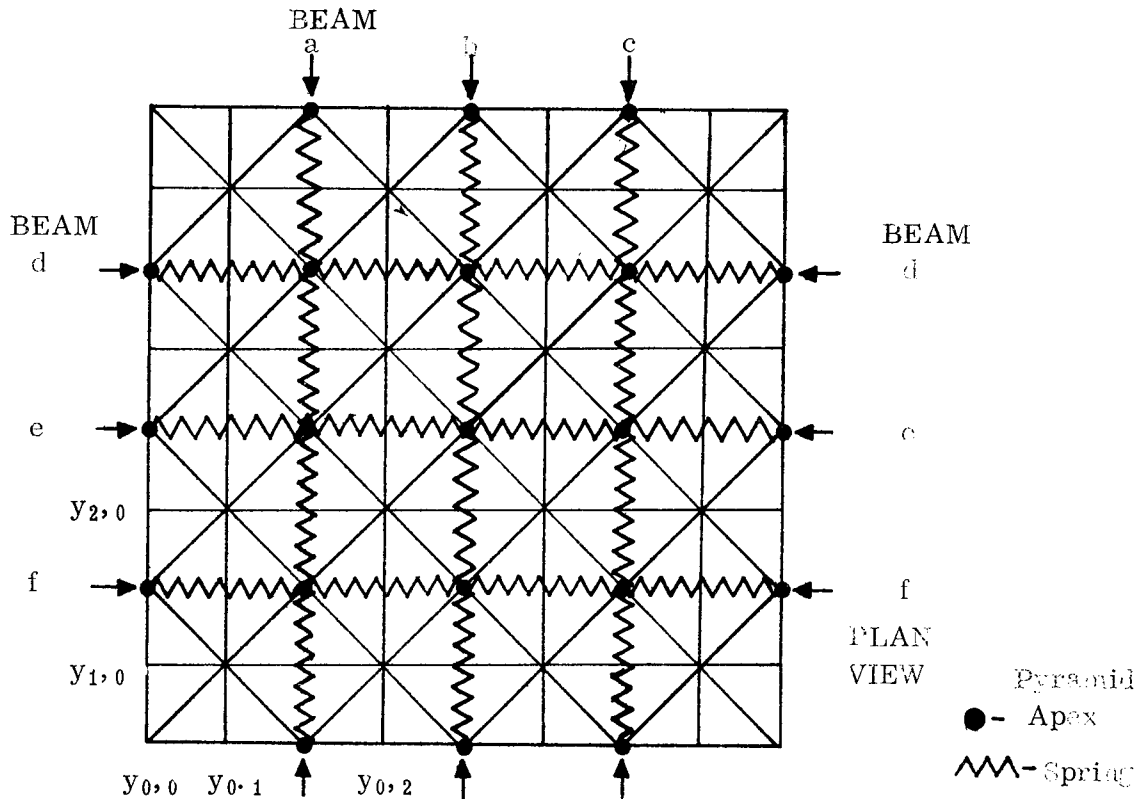
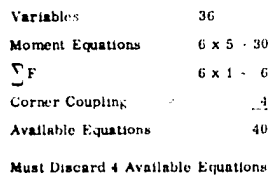


FIGURE 24. FOUR SPRING PLATE MODEL

There are three horizontal beams and three vertical beams. The four corners of the plate are not a part of any beam. However, when the assumption that the pyramids are rigid is applied, the three hinges of the corner segments that are attached to beams will determine the displacement of the corners.

Because of the coupling, the number of displacement variables of the plate is not equal to the sum of the six beams and the four corners. This fact reduces the number of required equations to make up a plate matrix equation. Each beam model must include all of the variables of that beam. When fewer equations are required, the fifth and sixth lines of the equation may be deleted and line seven substituted as the fifth equation of the beam set. The plate of Figure 24 will require a rectangular matrix of 60 by 36 which reduces to 36 by 36 when the end conditions are inserted. Equation 40 gives the analytical model of the plate. The end conditions are selected in the same manner as the single beams and all the variations as described for the beams may be applied to the plate matrix.

The stiffness constant of the two dimensional beam model is



1

[illegible]

[illegible]

[illegible]





0	0	0	0	0	0	0	0	Y00	R <sub>no</sub>
0	0	0	0	0	0	0	0	Y01	R <sub>bo</sub>
0	0	0	0	0	0	0	0	Y02	R <sub>co</sub>
0	0	0	0	0	0	0	0	Y03	R <sub>do</sub>
0	-I <sup>-1</sup>	0	0	0	0	0	0	Y04	R <sub>eo</sub>
0	0	0	0	0	0	0	0	Y05	R <sub>fo</sub>
0	0	0	0	0	0	0	0	Y10	R <sub>ns</sub>
0	0	0	0	0	0	0	0	Y11	R <sub>bs</sub>
0	0	0	0	0	0	0	0	Y12	R <sub>cs</sub>
0	0	0	0	0	0	0	0	Y13	R <sub>ds</sub>
0	0	-I <sup>-1</sup>	0	0	0	0	0	Y14	R <sub>es</sub>
0	0	0	0	0	0	0	0	Y15	R <sub>fs</sub>
0	0	0	0	0	0	0	0	Y20	-M <sub>no</sub>
0	0	0	0	0	0	0	0	Y21	-M <sub>bo</sub>
0	0	0	0	0	0	0	0	Y22	-M <sub>co</sub>
0	0	0	-I <sup>-1</sup>	0	0	0	0	Y23	-M <sub>do</sub>
0	0	0	0	0	0	0	0	Y24	-M <sub>eo</sub>
-I <sup>-1</sup>	0	0	0	0	0	0	0	Y25	-M <sub>fo</sub>
-I <sup>-1</sup>	0	0	0	0	0	0	0	Y30	-M <sub>ns</sub>
-I <sup>-1</sup>	0	0	0	0	0	0	0	Y31	-M <sub>bs</sub>
0	0	0	0	0	0	0	-I <sup>-1</sup>	Y32	-M <sub>cs</sub>
0	0	0	0	0	0	0	0	Y33	-M <sub>ds</sub>
0	0	0	0	0	0	0	0	Y34	-M <sub>es</sub>
0	0	0	0	0	0	0	0	Y35	-M <sub>fs</sub>
0	0	0	0	0	0	0	0	Y40	
0	0	0	0	0	0	0	0	Y41	
0	0	0	0	0	0	-I <sup>-1</sup>	0	Y42	
0	0	0	0	0	0	0	0	Y43	
0	0	0	0	0	0	0	0	Y44	
0	0	0	0	0	0	0	0	Y45	
0	0	0	0	0	0	0	0	Y50	
0	0	0	0	-I <sup>-1</sup>	0	0	0	Y51	
0	0	0	0	0	0	0	0	Y52	
0	0	0	0	0	0	0	0	Y53	
0	0	0	0	0	0	0	0	Y54	
0	0	0	0	0	0	0	0	Y55	

Continued

Equation (40)



$$K = \frac{E I}{\ell^3 (1-\nu^2)}, \quad (41)$$

where  $\nu$  is Poisson's Ratio and the other nomenclature is the same as in the beam.

The 4 by 4 spring plate solutions will be accurate for only the first mode of the plate. Practical considerations would dictate a matrix up to the capacity of the available computer. For example a plate with 13 by 13 springs and good for 3 modes in each direction would require a 169 by 169 matrix; a plate good for 4 modes in each direction would require a 289 by 289 matrix.

Since maximum amplitudes will be encountered in the lower modes, there is practical value from the use of the model. The Eigenvalue problems need only to be computed for a limited number of the lowest frequencies. Higher frequency solutions would have little meaning.

## SECTION VI. VARIATIONS OF THE PLATE MODEL

There are two major variations of the plate model that have significance. There are adaptations of the model to allow the inclusion of stiffening beams, and the simulation of a cut-out plate for a complex beam structure. Both represent a use of additions or subtractions of springs and masses to the plate model. The matrix equation can be modified to meet the specific applications.

### Shell Model

The analytical model can be modified to simulate a thin cylindrical shell structure. The deflection is assumed in a radial direction. With  $r$   $\phi$  substituted into the matrix equation for  $\ell$  of the wraparound beam, the wraparound beams are continuous and therefore have no end conditions. This changes the number of unknowns and also the character of available equations to set up a suitable matrix.

The adaptation of analytical principles used in this paper will be extended to cylinders and published at a later date.

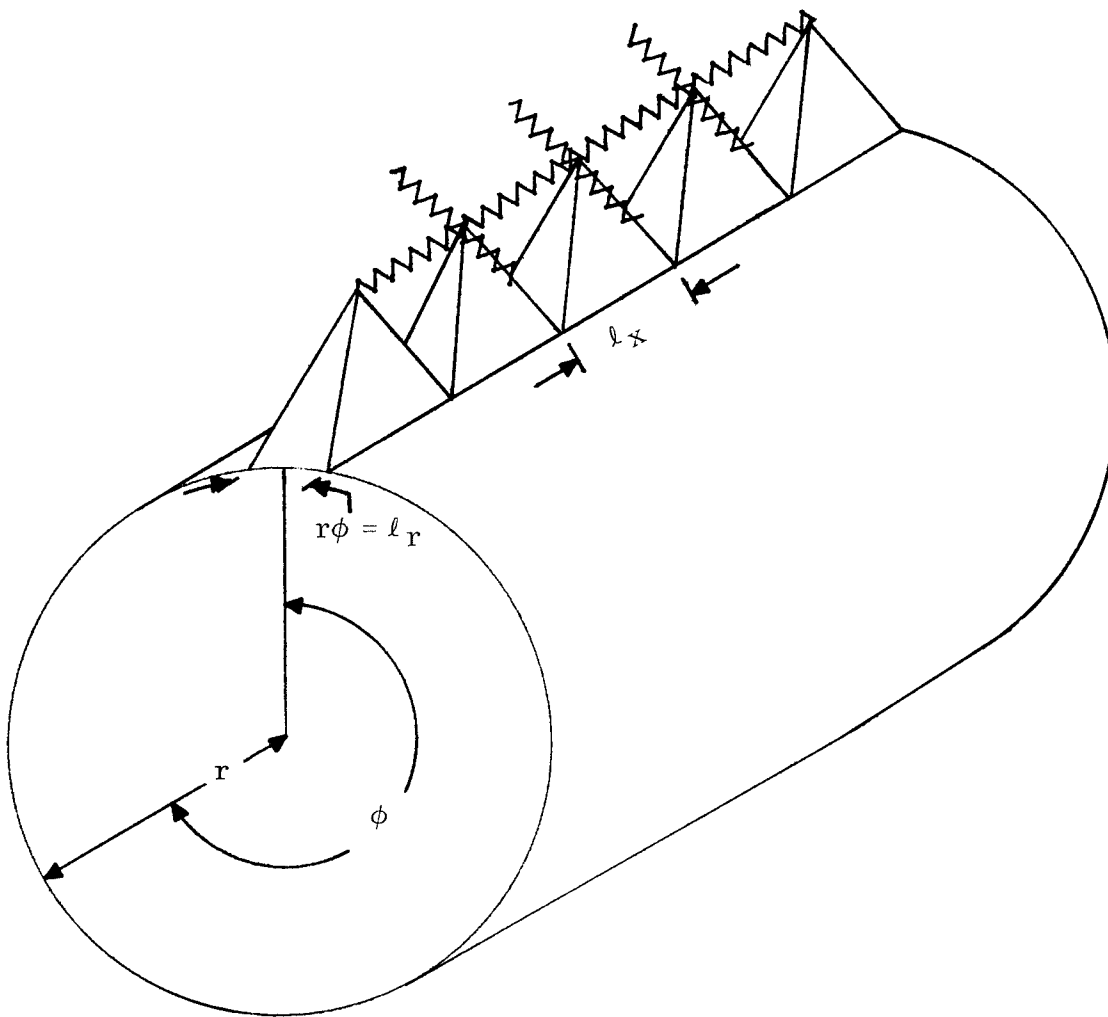


FIGURE 25. CYLINDRICAL SHELL MODEL

## SECTION VII. CONCLUSIONS

The dynamics of a beam or plate can be simulated by a model made up of massless constraints, springs, and concentrated masses. By unique arrangement of the mechanical elements, a mathematical model may be produced for structures not generally solvable by usual techniques. The analytical approach has been derived and the results proven by the use of simple hand-solvable models. Extension to more complicated structures is possible using digital computers for matrix Eigenvalue and Eigenvector solutions.

George C. Marshall Space Flight Center  
National Aeronautics and Space Administration  
Huntsville, Alabama, February 1965